

ON SINGULAR EQUATIONS WITH CRITICAL AND SUPERCRITICAL EXPONENTS

MOUSOMI BHAKTA AND SANJIBAN SANTRA

ABSTRACT. We study the problem

$$(I_\varepsilon) \begin{cases} -\Delta u - \frac{\mu u}{|x|^2} = u^p - \varepsilon u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^{q+1}(\Omega), \end{cases}$$

where $q > p \geq 2^* - 1$, $\varepsilon > 0$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $0 \in \Omega$, $N \geq 3$ and $0 < \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$. We completely classify the singularity of solution at 0 in the supercritical case. Using the transformation $v = |x|^\nu u$, we reduce the problem (I_ε) to (J_ε)

$$(J_\varepsilon) \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla v) = |x|^{-(p+1)\nu} v^p - \varepsilon |x|^{-(q+1)\nu} v^q & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v \in H_0^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu}), \end{cases}$$

and then formulating a variational problem for (J_ε) , we establish the existence of a variational solution v_ε and characterise the asymptotic behaviour of v_ε as $\varepsilon \rightarrow 0$ by variational arguments and when $p = 2^* - 1$.

This is the first paper where the results have been established with super critical exponents for $\mu > 0$.

CONTENTS

1. Introduction	2
2. Existence and non-existence of entire solution	11
3. Classification of singularity near 0	14
3.1. Lower and upper estimate of solution	14
3.2. The Critical Case $q = \frac{2+\nu}{\nu}$	23
3.3. Gradient estimate	24
4. Holder continuity and Green function estimates	26
5. Symmetry and decay properties of entire problem	31
6. Proof of Theorem 1.10	34
6.1. Auxiliary results	34
6.2. Asymptotic Behavior	37
7. The case $p = 2^* - 1$ and proof of Theorem 1.11	40
Appendix A.	48
Appendix B.	50
References	51

2010 *Mathematics Subject Classification.* Primary 35B08, 35B40, 35B44.

Key words and phrases. super-critical exponent, Hardy's inequality, local estimates, gradient estimates, asymptotic behaviour, entire solution, blow-up, Green function estimates, large solutions.

1. INTRODUCTION

In this paper, we consider the following family of singular problems:

$$(1.1) \quad \begin{cases} -\Delta u - \frac{\mu u}{|x|^2} = u^p - \varepsilon u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^{q+1}(\Omega), \end{cases}$$

and

$$(1.2) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla v) = |x|^{-(p+1)\nu} v^p - \varepsilon |x|^{-(q+1)\nu} v^q & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v \in H_0^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu}), \end{cases}$$

where $q > p \geq 2^* - 1 = \frac{N+2}{N-2}$, $\varepsilon > 0$ is a parameter, $\Omega \subseteq \mathbb{R}^N$ is a star-shaped bounded domain with smooth boundary, $0 \in \Omega$, $N \geq 3$ and $0 < \mu < \bar{\mu} := (\frac{N-2}{2})^2$ and $\nu \in (0, \frac{N-2}{2})$. By the Pohozaev's identity, we know that when $\varepsilon = \mu = \nu = 0$ and Ω is star shaped, (1.1) and (1.2) have no solutions. In this paper, we are mainly concerned with two issues. One of them is to classifying the nature of singularity to the solutions of Eq.(1.1) and the other one is to study the asymptotic behavior of solutions of the problem (1.2) as $\varepsilon \rightarrow 0$. When $\mu = 0$, the asymptotic behaviour of this class of equation with supercritical exponent was studied by Merle and Peletier in [18, 19]. Also see McLeod *et.al.* [17] for the uniqueness proof for the entire solution in the supercritical case, Han [14] and Brezis–Peletier[2] for the subcritical blow up. As per our knowledge, there is no existing result with supercritical exponents for $\mu > 0$.

We assume that

$$(1.3) \quad v_\varepsilon(0) = \max_{\Omega} v_\varepsilon(x).$$

If we look for radial solutions of Eq. (1.1), we would expect u as a function of the radial variable r to behave like Ar^{-m} near 0, where A and m satisfy

$$(1.4) \quad A[-m(m+1) + m(N-1) - \mu]r^{-m-2} = -(1+o(1))A^q r^{-mq},$$

so that either

$$(1.5) \quad m(m-N+2) + \mu > 0, \quad m+2 = mq \implies m = \frac{2}{q-1} \quad \text{and} \quad q > \frac{\mu+2\nu'}{\mu}$$

or

$$(1.6) \quad m+2 < mq, \quad m(m+1) - m(N-1) + \mu = 0 \implies m = \nu \text{ or } \nu',$$

for

$$(1.7) \quad \nu := \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}; \quad \nu' := \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}.$$

Note that $\nu < \nu'$. Also, one can readily check $\frac{\mu+2\nu'}{\mu} = \frac{2+\nu}{\nu}$. In the region where $q < \frac{2+\nu}{\nu}$, we have $\nu < \frac{2}{q-1}$. Therefore in this region

$$(1.8) \quad r^{-\nu} < c \min\{r^{-\frac{2}{q-1}}, r^{-\nu'}\}.$$

On the other hand, in the region where $q \geq \frac{2+\nu}{\nu}$ we have

$$(1.9) \quad r^{-\frac{2}{q-1}} \leq c \min\{r^{-\nu}, r^{-\nu'}\}.$$

It is easy to check from (1.4) that for the blow up with exponent $\frac{2}{q-1}$ (see (1.5)) the constant A would be determined, whereas for the second type of blow up it would appear to be free.

In Section 3 we prove that near 0, any solution u of Eq.(1.1) satisfies

$$C_1|x|^{-\nu} \leq u(x) \leq C_2|x|^{-\nu} \quad \text{if } 2^* - 1 \leq p < q < \frac{2+\nu}{\nu},$$

and

$$C_3|x|^{-\frac{2}{q-1}} \leq u(x) \leq C_4|x|^{-\frac{2}{q-1}} \quad \text{if } q > \max \left\{ p, \frac{2+\nu}{\nu} \right\},$$

for some positive constants C_1, C_2, C_3 and C_4 . Moreover when $u(x) = u(|x|)$ and $q = \frac{2+\nu}{\nu}$,

$$u(|x|) \sim |x|^{-\nu} |\log |x||^{-\frac{\nu}{2}}, \quad \text{as } |x| \rightarrow 0.$$

More precisely, if

$$-\Delta u - \frac{\mu u}{|x|^2} = f_i(u), \quad i = 1, 2,$$

we can classify the singularity of $u(x)$ near the origin with the nonlinearities $f_1(u) = u^p + \varepsilon u^q$ where $1 \leq q < p = 2^* - 1$ or $f_2(u) = u^p - \varepsilon u^q$ where $2^* - 1 \leq p < q$ in the following way.

Range of (p, q)	$1 \leq q < p = 2^* - 1$	$2^* - 1 \leq p < q < \frac{2}{\nu} + 1$	$q > \max\{p, \frac{2}{\nu} + 1\}$
Singularity at 0	$C_1 \leq x ^\nu u(x) \leq C_2$	$C_1 \leq x ^\nu u(x) \leq C_2$	$C_1 \leq x ^{\frac{2}{q-1}} u(x) \leq C_2$

for some $C_1 > 0, C_2 > 0$. For the subcritical case see [4, 13].

Near 0, Eq.(1.1) can be written as $-\Delta u - \mu \frac{u}{|x|^2} = -(1 + o(1))u^q$. Therefore, if u is radial then by setting $v(r) = r^\nu u(r)$, the above equation reduces to

$$(1.10) \quad v'' + \frac{N-1-2\nu}{r} v' = A r^{-(q-1)\nu} v^q \quad \text{in } (0, a),$$

for some $a > 0$ and $1 - \delta < A < 1 + \delta$, for some $\delta > 0$ small. Using the Emden-Fowler transformation $t = (\frac{\alpha}{r})^\alpha$ and $y(t) = \alpha^{-\nu} v(r)$, where $\alpha = N - 2 - 2\nu$, (1.10) reduces to the so-called Emden-Fowler type equation

$$y''(t) = A t^{\frac{-(2\alpha+2)+(q-1)\nu}{\alpha}} y^q, \quad t \geq R,$$

for some $R > 0$ large. These type of equations have several interesting applications in mathematical physics. It appears in astrophysics in the form of Emden equation and in atomic physics in the form of Thomas-Fermi statistical model of atoms. Emden-Fowler type equations appears in modelling the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. For more details see [1, 10, 15, 21].

Recently, a great deal of attention is given to the mathematical study of following class of semilinear elliptic problem

$$(1.11) \quad \begin{cases} \Delta u + \frac{\mu}{|x|^2} u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a super-linear function; $0 \in \Omega$ is a smooth bounded domain in \mathbb{R}^N , $0 \leq \mu < \bar{\mu} = \frac{(N-2)^2}{4}$ and $N \geq 3$. This class of problems is of particular interest

as this arises in mathematical models related to reaction diffusion equations and celestial mechanics. We recall the classical Hardy's inequality: if $u \in H_0^1(\Omega)$, then

$$(1.12) \quad \int_{\Omega} |\nabla u|^2 dx \geq \bar{\mu} \int_{\Omega} \frac{u^2}{|x|^2} dx,$$

where $\bar{\mu}$ is never achieved by any $u \in H_0^1(\Omega)$.

We denote by $D^{1,2}(\mathbb{R}^N)$, the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{\frac{1}{2}}$. When $0 < \mu < \bar{\mu}$, it is easy to check that the following is an equivalent norm for $D^{1,2}(\mathbb{R}^N)$:

$$(1.13) \quad \|u\|_{D^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \right)^{\frac{1}{2}}.$$

When $\mu = 0$, define

$$(1.14) \quad \mathcal{S}(v) := \frac{\int_{\Omega} |\nabla v|^2 dx}{\left(\int_{\Omega} v^{2^*} dx \right)^{\frac{2}{2^*}}}, \quad \mathcal{S}_N = \inf_{v \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 dx}{\left(\int_{\mathbb{R}^N} v^{2^*} dx \right)^{\frac{2}{2^*}}}.$$

For $N \geq 3$ and $\nu = 0$, Merle–Peletier [19] proved :

Theorem(Merle–Peletier, [19])

- (i) There exist ε and θ_ε with $\varepsilon \rightarrow 0$ and θ_ε is uniformly above and away from 0, such that there exists a solution u_ε of Eq. (1.2) with $\nu = 0$ and Furthermore, if $p = 2^* - 1$, then $S(\theta_\varepsilon u_\varepsilon) \rightarrow \mathcal{S}_N$ as $\varepsilon \rightarrow 0$ and there exists constants A, B such that $A < \int_{\Omega} u_\varepsilon^{p+1} < B$.

if $p > 2^* - 1$, then $K(\theta_\varepsilon u_\varepsilon) \rightarrow K_N$ as $\varepsilon \rightarrow 0$ and $\int_{\Omega} u_\varepsilon^{p+1} dx \rightarrow 0$ as $\varepsilon \rightarrow 0$, where

$$K(u) = \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^{p+1}} + \frac{\int_{\Omega} |u|^{q+1}}{\left(\int_{\Omega} |u|^{p+1} \right)^l}; \quad l = \frac{2(q+1) - N(p-1)}{2(p+1) - N(p-1)},$$

and

$$K_N = \inf \left\{ K(u) : u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \int_{\Omega} |u|^{p+1} = 1 \right\}.$$

- (ii) Let x_ε be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_\infty$ and assume that up to a subsequence $x_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$. Then in the case $p = 2^* - 1$,

$$\varepsilon^{\frac{1}{q-p+2}} \|u_\varepsilon\|_\infty \sim A(q, N) \quad \text{as } \varepsilon \rightarrow 0.$$

and when $x \neq 0$

$$\varepsilon^{-\frac{1}{q-p+2}} u_\varepsilon(x) \rightarrow [N(N-2)]^{\frac{N-2}{2}} \frac{G(x, x_0)}{A(q, N)R(x_0)} \quad \text{as } \varepsilon \rightarrow 0.$$

where

$$A(q, N) = \left[\frac{N^2 c(q, N)}{[N(N-2)]^{\frac{N}{2}}} B \left(\frac{N}{2}, q \frac{N-2}{2} - 1 \right) \right]; \quad c(q, N) = \frac{(N-2)q - (N+2)}{2(q+1)}$$

and $B(a, b)$ denotes the Beta function [19] defined by

$$(1.15) \quad B(a, b) = \int_0^\infty t^{a-1} (1+t)^{-a-b} dt.$$

G is the Green function and x_0 is the critical point of the Robin function, see (1.27) with $\nu = 0$. Moreover, if $p > 2^* - 1$, then

$$\varepsilon^{\frac{1}{q-p}} \|u_\varepsilon\|_\infty \sim c^* \quad \text{as } \varepsilon \rightarrow 0,$$

and when $x \neq 0$,

$$(1.16) \quad \varepsilon^{-\theta} u_\varepsilon(x) \rightarrow (c^*)^{-\theta} (J_p - c^* J_q) G(x, x_0), \quad \text{as } \varepsilon \rightarrow 0,$$

where

$$\theta = \frac{(N-2)p - N}{2(q-p)}; \quad J_p = \int_{\mathbb{R}^N} V^p dx$$

and (c^*, V) is the unique solution of

$$\begin{cases} -\Delta V = V^p - c^* V^q & \text{in } \mathbb{R}^N, \\ V \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) & . \end{cases}$$

In [16, 20] the following problem with critical exponent and Hardy potential was studied:

$$(1.17) \quad \begin{cases} -\Delta u - \frac{\mu}{|x|^2} u = u^{2^*-1} + \varepsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \in \Omega \subset \mathbb{R}^N$; $N \geq 3$, $0 < \mu < \bar{\mu}$ and $\varepsilon > 0$ is a parameter. Jannelli [16] proved the following: If $0 \leq \mu \leq \bar{\mu} - 1$, then (1.17) has a positive solution $u \in H_0^1(\Omega)$ for all $0 < \varepsilon < \lambda_1$. Furthermore, he proved that if $\bar{\mu} - 1 < \mu < \bar{\mu}$, Eq.(1.17) has a positive solution $u \in H_0^1(\Omega)$ if and only if $\varepsilon \in (\lambda^*, \lambda_1)$ for some $\lambda^* \in (0, \lambda_1)$, when Ω is the ball then Eq.(1.17) has no positive solution for all $\varepsilon \leq \lambda^*$. Cao-Peng [4] studied problem similar to Eq.(1.17) for the almost critical case. Cao-Peng [4] and Ramaswamy-Santra [20] used the radial nature of the positive solution to obtain the global uniqueness and blow-up profile as $\varepsilon \rightarrow 0$. It was proved in [20], when $N \geq 5$ and $v_\varepsilon \in H_0^1(\Omega, |x|^{-2\nu})$ is a solution of Eq. (1.17) satisfying

$$\mathcal{S} = \lim_{\varepsilon \rightarrow 0} \frac{\| |x|^{-\nu} |\nabla v_\varepsilon| \|_{L^2(\Omega)}}{\| |x|^{-\nu} v_\varepsilon \|_{L^{2^*}(\Omega)}^2}, \quad \mu < \bar{\mu} - 1,$$

then along a subsequence

$$(1.18) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \|v_\varepsilon\|_\infty^{\frac{2(N-2\nu-4)}{N-2-2\nu}} = \frac{(N-2)^2}{2N^2(N-2-2\nu)b_n} \mathcal{S}(\mu)^{-\frac{N}{2}} \sigma_N |R(0)|$$

where $b_n = \int_0^\infty \frac{t^{N-2\nu-1}}{(1+t^{\frac{2\alpha}{N-2}})^{N-2}} dt$; and when $x \neq 0$

$$\lim_{\varepsilon \rightarrow 0} v_\varepsilon(x) \|v_\varepsilon\|_\infty = \frac{N-2}{N(N-2-2\nu)} \omega_N G(x, 0),$$

where $R(0)$ and $G(x, 0)$ are as defined in (1.27) and (1.24) respectively.

Define,

$$(1.19) \quad \mathcal{S} = \inf_{u \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |x|^{-2^* \nu} u^{2^*} dx \right)^{\frac{2}{2^*}}},$$

where $\nu \in \left[0, \frac{N-2}{2}\right)$. Thanks to Caffarelli-Kohn-Nirenberg inequality [3], we have $\mathcal{S} > 0$. It is also well-known that \mathcal{S} in the above expression is same as

$$\inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx}{\left(\int_{\Omega} u^{2^*} dx \right)^{\frac{2}{2^*}}}$$

and independent of the domain Ω , where $\mu \in (0, \bar{\mu})$. In the above two expression of \mathcal{S} , the parameters μ and ν are related by $\nu = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$. From [5], we know

$$\mathcal{S} = \mathcal{S}_N \left(1 - \frac{4\mu}{(N-2)^2} \right)^{\frac{N-1}{N}},$$

where \mathcal{S}_N is the usual Sobolev constant. Moreover, Catrina-Wang [5, 9] proved that \mathcal{S} is achieved by

$$(1.20) \quad U(x) = \left(\frac{N\alpha^2}{N-2} \right)^{\frac{N-2}{4}} (1 + |x|^{\frac{2\alpha}{N-2}})^{-\frac{N-2}{2}},$$

where

$$(1.21) \quad \alpha = N - 2 - 2\nu.$$

Furthermore, by [22], U is the unique solution (dilation invariant) of the following entire problem:

$$(1.22) \quad \begin{cases} -\nabla(|x|^{-2\nu} \nabla U) = |x|^{-2^* \nu} U^{2^*-1} & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N \setminus \{0\}, \\ U \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}). \end{cases}$$

Define the Green's function G as

$$(1.23) \quad H(x, y) = G(x, y) + F(x, y),$$

where $G(x, y)$ is defined by

$$(1.24) \quad \begin{cases} \nabla_x(|x|^{-2\nu} \nabla_x G(x, y)) = \delta_y & \text{in } \Omega, \\ G(x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

and $H(x, y)$ is the regular part of the Green function

$$(1.25) \quad \begin{cases} \nabla_x(|x|^{-2\nu} \nabla_x H(x, y)) = 0 & \text{in } \Omega, \\ H(x, y) = F(x, y) & \text{on } \partial\Omega, \end{cases}$$

for any fixed $y \in \Omega$ and

$$(1.26) \quad F(x, y) = -\frac{1}{(N-2-2\nu)\omega_N |x-y|^{N-2\nu-2}}$$

is the fundamental solution of the non-degenerate elliptic operator $\nabla(|x|^{-2\nu}\nabla)$. By construction, $H(x, 0)$ is negative and Hölder continuous near the origin [6]. Define the Robin function as

$$(1.27) \quad R(x) = H(x, x).$$

Hence R is continuous at the origin and we can write

$$\lim_{|x| \rightarrow 0} R(x) = R(0).$$

For the supercritical case ($p > 2^* - 1$), we define the functional

$$(1.28) \quad F(v, \Omega) = \frac{1}{2} \frac{\int_{\Omega} |x|^{-2\nu} |\nabla v|^2 dx}{\int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx} + \frac{1}{q+1} \frac{\int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx}{\left(\int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx \right)^l},$$

where

$$(1.29) \quad l = \frac{2(q+1) - N(p-1)}{2(p+1) - N(p-1)},$$

and $v \in H_0^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu})$.

Also define,

$$(1.30) \quad \mathcal{K} := \inf \left\{ F(v, \mathbb{R}^N) : v \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu}), \int_{\mathbb{R}^N} |x|^{(p+1)\nu} |v|^{p+1} = 1 \right\}.$$

For the critical case ($p = 2^* - 1$), we consider the usual functional

$$(1.31) \quad S(v) = \frac{\int_{\Omega} |x|^{-2\nu} |\nabla v|^2 dx}{\left(\int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx \right)^{\frac{2}{p+1}}},$$

where $v \in H_0^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu})$.

We turn now to a brief description of the results presented below. The first result concerns the non-existence result when $p = 2^* - 1$.

Theorem 1.1. *Let $0 \leq \mu < \bar{\mu}$ and*

$$(1.32) \quad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = u^{2^*-1} - u^q & \text{in } \mathbb{R}^N, \\ u > 0, \\ u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \end{cases}$$

where $q > 2^* - 1$. Then Eq.(1.32) does not admit any solution.

The proof of this theorem is based on the Pohozaev identity. The difficulty in applying this identity comes from the fact that the solution blows up at origin (see Section 3).

Setting the transformation $v = |x|^\nu u$ in (1.28) and (1.30) we obtain

$$(1.33) \quad F(u, \Omega) = \frac{1}{2} \frac{\int_{\Omega} \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) dx}{\int_{\Omega} u^{p+1} dx} + \frac{1}{q+1} \frac{\int_{\Omega} u^{q+1} dx}{\left(\int_{\Omega} u^{p+1} dx \right)^l},$$

where $p > 2^* - 1$ and l is same as in (1.29), $u \in H_0^1(\Omega) \cap L^{q+1}(\Omega)$ (see [13, Theorem 1.1]) and

$$(1.34) \quad \mathcal{K} := \inf \left\{ F(u, \mathbb{R}^N) : u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{p+1} = 1 \right\}.$$

Theorem 1.2. *Let $0 \leq \mu < \bar{\mu}$, $N \geq 3$ and $q > p > 2^* - 1$. Then \mathcal{K} in (1.34) is achieved by a radially decreasing function in $D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$. Furthermore, there exists a constant $\lambda > 0$*

$$(1.35) \quad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \lambda u^p - u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N). \end{cases}$$

Theorem 1.3. *Assume $2^* - 1 \leq p < q < \frac{2+\nu}{\nu}$ and u be any solution (whenever exists) of*

$$(1.36) \quad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = u^p - u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^{q+1}(\Omega), \end{cases}$$

with $0 < \mu < \bar{\mu}$ and Ω be any smooth domain (bounded or unbounded). Then there exists $r_0 > 0$ (small) and $C_1 > 0$ (r_0 and C_1 independent of u) such that u satisfies

$$u(x) \geq C_1 |x|^{-\nu} \quad \forall \quad x \in B_{r_0}(0) \setminus \{0\}.$$

Remark 1.1. *Standard methods of finding lower estimate, e.g. the methods of [4, 13] do not work here. In Section 3, we have shown that to get the estimate $|u(x)| \geq C|x|^{-\nu}$, it is enough to show that solution of the following equation is bounded away from 0,*

$$-\operatorname{div}(|x|^{-2\nu} \nabla w) + |x|^{-(q+1)\nu} w^q = 0.$$

To show the existence of positive solution of this equation with suitable boundary data and which is bounded away from 0, we have used ODE technique, Banach fixed point theorem and comparison principle.

Theorem 1.4. (i) *If $p = 2^* - 1$, then any solution u of Eq. (1.36) satisfies*

$$u(x) \leq C|x|^{-\nu} \quad \forall \quad x \in B_{\rho_0}(0) \setminus \{0\},$$

where $\rho_0 > 0$ is sufficiently small.

(ii) *If $p > 2^* - 1$ and $q > (p-1)\frac{N}{2} - 1$ then the same conclusion holds as in (i).*

Remark 1.2. *Since $p = 2^* - 1$ implies $(p-1)\frac{N}{2} - 1 = 2^* - 1$, the condition $q > (p-1)\frac{N}{2} - 1$ is readily satisfied in the case $p = 2^* - 1$ as q is supercritical.*

Theorem 1.5. *Let $\mu \in (0, \bar{\mu})$ and $q > \max\{p, \frac{2+\nu}{\nu}\}$. Then any solution of Eq.(1.36) satisfies*

$$u(x) \leq C|x|^{-\frac{2}{q-1}} \quad \forall \quad x \in B_\rho(0) \setminus \{0\},$$

where $\rho > 0$ is sufficiently small.

Theorem 1.6. *Let $\mu \in (0, \bar{\mu})$ and $q > \max\{p, \frac{2+\nu}{\nu}\}$. Then any solution of Eq.(1.36) satisfies*

$$u(x) \geq C|x|^{-\frac{2}{q-1}} \quad \forall \quad x \in B_R(0) \setminus \{0\},$$

where $R > 0$ is sufficiently small.

Theorem 1.7. *Let $\mu \in (0, \bar{\mu})$ and $q = \frac{2+\nu}{\nu}$. Then any radial solution u of Eq.(1.36) satisfies*

$$\lim_{|x| \rightarrow 0} \frac{u(x)}{|x|^{-\nu} |\log |x||^{-\frac{\nu}{2}}} = \left(\frac{\alpha\nu}{2}\right)^{\frac{\nu}{2}}.$$

where $\alpha = N - 2 - 2\nu$.

Theorem 1.8. *Let $2^* - 1 \leq p \leq (p-1)\frac{N}{2} - 1 < q$ and $\tilde{\rho} := \frac{1}{2} \min\{\rho_0, \rho\}$, where ρ_0 and ρ be as in Theorem 1.4 and Theorem 1.5 respectively. Then there exists $\mu^* = \mu^*(N, q) > 0$ and a constant C depending on N, p, q, μ such that any solution u of Eq. (1.36) satisfies*

$$|\nabla u(x)| \leq \begin{cases} C|x|^{-(\nu+1)} & \text{if } \mu \in [0, \mu^*), \\ C|x|^{-(\frac{q+1}{q-1})} & \text{if } \mu \in [\mu^*, \bar{\mu}), \end{cases}$$

for $0 < |x| < \tilde{\rho}$.

Remark 1.3. *In the above theorem $\mu^* = \left(\frac{N-2}{2}\right)^2 - \left(\frac{N-2}{2} - \frac{2}{q-1}\right)^2$. It's easy to note that $\mu < \mu^* \iff q < \frac{2+\nu}{\nu}$. From Theorem 1.3 and Theorem 1.4, it follows any solution u has singularity of the order ν when $q < \frac{2+\nu}{\nu}$. Therefore in this region of q , it is anticipated that $|\nabla u| \leq C|x|^{-(\nu+1)}$ near 0. On the other hand when $q > \frac{2+\nu}{\nu}$, from Theorem 1.5 and Theorem 1.6, we have singularity of u at 0 is of order $\frac{2}{q-1}$. Consequently in this region we expect $|\nabla u| \leq C|x|^{-(\frac{q+1}{q-1})}$.*

Theorem 1.9. *Let $2^* - 1 \leq p \leq (p-1)\frac{N}{2} - 1 < q$ and $0 \leq \mu < (\frac{N-2}{2})^2$. Then any positive solution $u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ of Eq. (1.35) is radially symmetric with respect to origin and radially decreasing.*

To discuss the asymptotic behaviour of problem (1.2) for general domain, we first formulate a variational problem for (1.2). Then we establish existence of variational solution v_ε for small positive values of ε and finally we derive the asymptotic behavior of v_ε as $\varepsilon \rightarrow 0$, using variational arguments again. This is the first result for the problem with critical and supercritical exponent in the singular case and the case appears to be more complicated than the smooth case.

Theorem 1.10. *There exists $\varepsilon_n > 0$ and $\lambda_n > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and λ_n uniformly bounded above and away from zero, such that*

- (i) *there exists a solution u_n to Eq. (1.2) corresponding to $\varepsilon = \varepsilon_n$;*
- (ii) *if $p > 2^* - 1$, then $F(\lambda_n u_n) \rightarrow \mathcal{K}$ and $\int_\Omega |x|^{-(p+1)\nu} u_n^{p+1} dx \rightarrow 0$ as $n \rightarrow \infty$;*

(iii) if $p = 2^* - 1$, then $S(\lambda_n u_n) \rightarrow \mathcal{S}$ as $n \rightarrow \infty$ and there exist constants

$$A, B > 0 \text{ such that for all } n \geq 1, \text{ it holds } A < \int_{\Omega} |x|^{-(p+1)\nu} u_n^{p+1} dx < B,$$

where $F(\cdot)$, \mathcal{K} and $S(\cdot)$, \mathcal{S} are defined as in (1.28), (1.30) and (1.31), (1.19) respectively.

Theorem 1.11. Let $\nu \in (0, \frac{N-2}{4})$, $2^* - 1 = p < q < \frac{1+\nu}{\nu}$ and $v_\varepsilon \in H_0^1(\Omega, |x|^{-2\nu})$ be a solution of Eq. (1.2) such that

$$S(\lambda_\varepsilon v_\varepsilon) \rightarrow \mathcal{S} \quad \text{and} \quad A < \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} dx < B,$$

where $S(\cdot)$, \mathcal{S} are as in (1.31) and (1.19) respectively. Moreover, assume (1.3) is satisfied. Then along a subsequence

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon \|v_\varepsilon\|_{\infty}^{\frac{q(N-2)-(N+2)+2\alpha}{\alpha}} \\ &= \frac{\omega_N |R(0)|}{C_{q,N}} \frac{(N-2-2\nu)}{(N-(q+1)\nu)} \frac{(N-2-4N\nu)}{\alpha} \frac{(N-(q+1)\nu)}{(N-2)} \frac{(N-2)-2\alpha(N-1)}{2\alpha} \\ & \quad \times \left[B \left(\frac{N-2}{2\alpha} (N-(q+1)\nu), \frac{N-2}{2\alpha} \{q(N-2-\nu) - (2+\nu)\} \right) \right]^{-1}, \end{aligned}$$

where

$$(1.37) \quad C_{q,N} = \frac{(N-2)q - (N+2)}{2(q+1)},$$

$R(0)$ and $B(a, b)$ are as defined in (1.27) and (1.15) respectively. Furthermore, for $x \neq 0$,

$$(1.38) \quad \lim_{\varepsilon \rightarrow 0} v_\varepsilon(x) \|v_\varepsilon\|_{\infty} = \omega_N (N-2-2\nu)^{N-1} \left(\frac{N}{N-2} \right)^{\frac{N-2}{2}} G(x, 0),$$

where $G(x, 0)$ is the Green function as defined in (1.24).

Remark 1.4. Now we point out the difference between the supercritical and subcritical case. First we notice there is a critical exponent $q^* := \frac{2+\nu}{\nu}$ which plays a huge role in determining the singularity of solution (1.1). This implies that there is some competition between the μ and q (or equivalently between ν and q) which never arise in the subcritical case.

Remark 1.5. In a forthcoming paper, we show this phenomena holds for the fractional laplacian case with $\mu = 0$.

Notation: Throughout this paper C denotes the generic constants which may vary from line to line. Below are few notations which we use throughout the paper:

- $\bar{\mu} := \left(\frac{N-2}{2} \right)^2$
- $\nu := \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$
- $\alpha := N - 2 - 2\nu$
- $\omega_N :=$ surface measure of unit ball.

2. EXISTENCE AND NON-EXISTENCE OF ENTIRE SOLUTION

In this section, we will study the existence and non-existence result of entire problem with critical and supercritical exponents. We first establish the general Pohozaev identity which will also be used in the next sections.

Proposition 2.1. *Let Ω be a smooth domain, $0 \in \Omega$, $0 \leq \mu < \bar{\mu}$, $N \geq 3$, $2^* - 1 \leq p < q$ and u be a solution of*

$$(2.1) \quad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = u^p - \varepsilon u^q & \text{in } \Omega, \\ u > 0, \\ u \in D^{1,2}(\Omega) \cap L^{q+1}(\Omega), \end{cases}$$

Then u satisfies:

$$(2.2) \quad \begin{aligned} & \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle x, n \rangle dS + \frac{N-2}{2} \int_{\partial\Omega} u \frac{\partial u}{\partial n} dS + \frac{\mu}{2} \int_{\partial\Omega} \frac{u^2}{|x|^2} \langle x, n \rangle dS \\ &= \frac{\varepsilon}{q+1} \int_{\partial\Omega} u^{q+1} \langle x, n \rangle dS - \varepsilon \left(\frac{N}{q+1} - \frac{N-2}{2} \right) \int_{\Omega} u^{q+1} dx \\ &+ \left(\frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} u^{p+1} dx - \frac{1}{p+1} \int_{\partial\Omega} u^{p+1} \langle x, n \rangle dS. \end{aligned}$$

In particular, if $u = 0$ on $\partial\Omega$ we have

$$(2.3) \quad \begin{aligned} & \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle x, n \rangle dS \\ &= \left(\frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} u^{p+1} dx + \varepsilon \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\Omega} u^{q+1} dx. \end{aligned}$$

Proof. We multiply Eq. (2.1) by a suitable test function and to make the test function smooth we introduce cut-off functions and then pass to the limit.

For $\delta > 0$ and $R > 0$, we define $\phi_{\delta,R}(x) = \phi_{\delta}(x)\psi_R(x)$ where $\phi_{\delta}(x) = \phi(\frac{|x|}{\delta})$ and $\psi_R(x) = \psi(\frac{|x|}{R})$, ϕ and ψ are smooth functions in \mathbb{R} with the properties $0 \leq \phi, \psi \leq 1$, with supports of ϕ and ψ in $(1, \infty)$ and $(-\infty, 2)$ respectively and $\phi(t) = 1$ for $t \geq 2$, and $\psi(t) = 1$ for $t \leq 1$.

Let u be a solution of Eq. (2.1). Then u is smooth away from the origin and hence $(x \cdot \nabla u)\phi_{\delta,R} \in C_c^2(\Omega)$. Multiplying Eq.(2.1) by this test function and integrating by parts, we obtain

$$(2.4) \quad \begin{aligned} & \int_{\Omega} \nabla u \cdot \nabla ((x \cdot \nabla u)\phi_{\delta,R}) dx - \mu \int_{\Omega} \frac{u(x \cdot \nabla u)\phi_{\delta,R}}{|x|^2} dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} (x \cdot \nabla u)\phi_{\delta,R} dS \\ &= \int_{\Omega} (u^p - \varepsilon u^q)(x \cdot \nabla u)\phi_{\delta,R} dx. \end{aligned}$$

Now the RHS of (2.4) can be simplified as

$$\begin{aligned} RHS &= -\frac{N}{p+1} \int_{\Omega} u^{p+1} \phi_{\delta,R} dx - \frac{1}{p+1} \int_{\Omega} u^{p+1} [x \cdot (\psi_R \nabla \phi_{\delta} + \phi_{\delta} \nabla \psi_R)] dx \\ &+ \varepsilon \frac{N}{q+1} \int_{\Omega} u^{q+1} \phi_{\delta,R} dx + \frac{\varepsilon}{q+1} \int_{\Omega} u^{q+1} [x \cdot (\psi_R \nabla \phi_{\delta} + \phi_{\delta} \nabla \psi_R)] dx \\ &+ \frac{1}{p+1} \int_{\partial\Omega} u^{p+1} \langle x, n \rangle \phi_{\delta,R} dS - \frac{\varepsilon}{q+1} \int_{\partial\Omega} u^{q+1} \langle x, n \rangle \phi_{\delta,R} dS. \end{aligned}$$

Note that $|x \cdot (\psi_R \nabla \varphi_\delta + \varphi_\delta \nabla \psi_R)| \leq C$ and hence using the dominated convergence theorem we get,

$$(2.5) \quad \begin{aligned} \lim_{R \rightarrow \infty} [\lim_{\delta \rightarrow 0} RHS] &= -\frac{N}{p+1} \int_{\Omega} u^{p+1} dx + \frac{N\varepsilon}{q+1} \int_{\Omega} u^{q+1} dx \\ &+ \frac{1}{p+1} \int_{\partial\Omega} u^{p+1} \langle x, n \rangle dS - \frac{\varepsilon}{q+1} \int_{\partial\Omega} u^{q+1} \langle x, n \rangle dS. \end{aligned}$$

By a direct calculation and integration by parts, LHS of (2.4) simplifies as,

$$(2.6) \quad \begin{aligned} \text{LHS} &= \int_{\Omega} |\nabla u|^2 \varphi_{\delta,R} + \sum_{i=1}^n \sum_{j=1}^n \frac{1}{2} \int_{\Omega} ((u_{x_i})^2)_{x_j} x_j \varphi_{\delta,R} + \int_{\Omega} (x \cdot \nabla u) (\nabla u \cdot \nabla \varphi_{\delta,R}) \\ &+ \frac{\mu N}{2} \int_{\Omega} \frac{u^2}{|x|^2} \varphi_{\delta,R} dx + \frac{\mu}{2} \int_{\Omega} \frac{u^2}{|x|^2} (x \cdot \nabla \varphi_{\delta,R}) dx - \mu \int_{\Omega} \frac{u^2}{|x|^2} \varphi_{\delta,R} dx \\ &- \int_{\partial\Omega} |\nabla u|^2 \langle x, n \rangle \phi_{\delta,R} dS - \frac{\mu}{2} \int_{\partial\Omega} \frac{u^2}{|x|^2} \langle x, n \rangle \phi_{\delta,R} dS \\ &= -\frac{N-2}{2} \left(\int_{\Omega} |\nabla u|^2 \varphi_{\delta,R} - \mu \int_{\Omega} \frac{u^2}{|x|^2} \varphi_{\delta,R} \right) dx - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle x, n \rangle \phi_{\delta,R} dS \\ &- \frac{1}{2} \int_{\Omega} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) [(x \cdot \nabla \varphi_{\delta}) \psi_R + (x \cdot \nabla \psi_R) \varphi_{\delta}] dx \\ &+ \int_{\Omega} (x \cdot \nabla u) [(\nabla u \cdot \nabla \varphi_{\delta}) \psi_R + (\nabla u \cdot \nabla \psi_R) \varphi_{\delta}] dx - \frac{\mu}{2} \int_{\partial\Omega} \frac{u^2}{|x|^2} \langle x, n \rangle \phi_{\delta,R} dS. \end{aligned}$$

Also we note that,

$$\begin{aligned} \lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \left| \int_{\mathbb{R}^N} |(x \cdot \nabla u) (\nabla u \cdot \nabla \varphi_{\delta}) \psi_R| dx \right| &\leq C \lim_{\delta \rightarrow 0} \int_{\delta \leq |x| \leq 2\delta} |\nabla u|^2 \frac{|x|}{\delta} dx \\ &\leq 2C \lim_{\delta \rightarrow 0} \int_{\delta \leq |x| \leq 2\delta} |\nabla u|^2 dx = 0. \end{aligned}$$

Similarly

$$\lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \left| \int_{\mathbb{R}^N} |(x \cdot \nabla u) (\nabla u \cdot \nabla \psi_R) \varphi_{\delta}| dx \right| \leq C \lim_{R \rightarrow \infty} \int_{R \leq |x| \leq 2R} |\nabla u|^2 \frac{|x|}{R} dx = 0.$$

Using the above estimates and taking the limit using dominated convergence theorem and using the fact $|x \cdot (\psi_R \nabla \varphi_{\delta} + \varphi_{\delta} \nabla \psi_R)| \leq C$, we get from (2.6),

$$(2.7) \quad \begin{aligned} \lim_{R \rightarrow \infty} [\lim_{\delta \rightarrow 0} LHS] &= -\frac{N-2}{2} \left(\int_{\Omega} |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx \\ &- \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle x, n \rangle dS - \frac{\mu}{2} \int_{\partial\Omega} \frac{u^2}{|x|^2} \langle x, n \rangle dS. \end{aligned}$$

Moreover, multiplying the Eq. (2.1) by u , we have

$$(2.8) \quad \int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} u (\nabla u \cdot n) dS - \mu \int_{\Omega} \frac{u^2}{|x|^2} dx = \int_{\Omega} (u^{p+1} - \varepsilon u^{q+1}) dx$$

Substituting (2.5) and (2.7) in (2.4) and using (2.8) we get

$$\begin{aligned}
& - \frac{N-2}{2} \left(\int_{\Omega} u^{p+1} dx - \varepsilon \int_{\Omega} u^{q+1} dx + \int_{\partial\Omega} u \frac{\partial u}{\partial n} dS \right) - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle x, n \rangle dS \\
& - \frac{\mu}{2} \int_{\partial\Omega} \frac{u^2}{|x|^2} \langle x, n \rangle dS = - \frac{N}{p+1} \int_{\Omega} u^{p+1} dx + \frac{N\varepsilon}{q+1} \int_{\Omega} u^{q+1} dx \\
(2.9) & + \frac{1}{p+1} \int_{\partial\Omega} u^{p+1} \langle x, n \rangle dS - \frac{\varepsilon}{q+1} \int_{\partial\Omega} u^{q+1} \langle x, n \rangle dS.
\end{aligned}$$

This implies (2.2). If $u = 0$ on $\partial\Omega$, it is easy to see that (2.3) follows from (2.2). \square

Proof of Theorem 1.1. If u is a solution of Eq.(1.32), then it follows from Proposition 2.1 that

$$\left(\frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} u^{q+1} dx = 0,$$

which is a contradiction as $q > 2^* - 1$ and $u > 0$. This proves the theorem. \square

Proof of Theorem 1.2. We are going to work on the manifold

$$N = \left\{ u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^{p+1} dx = 1 \right\}.$$

Then F reduces to

$$F(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu}{2} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \frac{1}{q+1} \int_{\mathbb{R}^N} u^{q+1} dx.$$

Let

$$(2.10) \quad \mathcal{K} = \inf_N F(u).$$

Let u_n be a minimizing sequence in N such that

$$F(u_n) \rightarrow \mathcal{K} \text{ with } \int_{\mathbb{R}^N} u_n^{p+1} dx = 1.$$

As $\mu < \bar{\mu}$ implies $\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right)^{\frac{1}{2}}$ is an equivalent norm in $D^{1,2}(\mathbb{R}^N)$, we have $\{u_n\}$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$ and $L^{q+1}(\mathbb{R}^N)$. Therefore there exists $u \in D^{1,2}(\mathbb{R}^N)$ and $L^{q+1}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and $L^{q+1}(\mathbb{R}^N)$. Consequently $u_n \rightarrow u$ pointwise almost everywhere.

Using symmetric rearrangement technique, without loss of generality we can assume that u_n is radially symmetric. Hence $u_n(x) = u_n(r)$, where $r = |x|$, and we can write

$$u_n(r) = - \int_r^\infty u'_n(s) ds.$$

Using a standard argument it can be shown that u_n satisfies Strauss type uniform estimate

$$(2.11) \quad |u_n(r)| \leq Cr^{-\frac{N-2}{2}}$$

for some $C > 0$. We claim that $u_n \rightarrow u$ in $L^{p+1}(\mathbb{R}^N)$.

To see the claim, we note that $u_n^{p+1} \rightarrow u^{p+1}$ pointwise almost everywhere. Since $\{u_n\}$ is uniformly bounded in $L^{q+1}(\mathbb{R}^N)$, using Vitali's convergence theorem, it is

easy to check that $\int_K u_n^{p+1} dx \rightarrow \int_K u^{p+1} dx$ for any compact set K in \mathbb{R}^N containing the origin. Furthermore, $\int_{\mathbb{R}^N \setminus K} u_n^{p+1} dx$ is very small by (2.11) and hence we have strong convergence. Moreover, $\int_{\mathbb{R}^N} u_n^{p+1} dx = 1$ implies $\int_{\mathbb{R}^N} u^{p+1} dx = 1$.

Now we show that $\mathcal{K} = F(u)$.

We note that $u \mapsto \|u\|^2$ is weakly lower semicontinuous. Therefore using Fatou's lemma we can write

$$\begin{aligned} \mathcal{K} &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \frac{\mu}{2} \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^2} dx + \frac{1}{q+1} \int_{\mathbb{R}^N} u_n^{q+1} dx \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \|u_n\|^2 + \frac{1}{q+1} \int_{\mathbb{R}^N} u_n^{q+1} dx \right] \\ &\geq \frac{1}{2} \|u\|^2 + \frac{1}{q+1} \int_{\mathbb{R}^N} u^{q+1} dx \\ &\geq F(u). \end{aligned}$$

This proves $F(u) = \mathcal{K}$. Moreover, using the Schwartz symmetrisation method via. Polya-Szego inequality, it is easy to check that u is radially symmetric and radially decreasing. Applying the Lagrange multiplier rule, we obtain u satisfies

$$-\Delta u - \mu \frac{u}{|x|^2} + u^q = \lambda u^p,$$

for some $\lambda > 0$. This in turn implies

$$-\Delta u - \mu \frac{u}{|x|^2} = \lambda u^p - u^q \quad \text{in } \mathbb{R}^N.$$

□

3. CLASSIFICATION OF SINGULARITY NEAR 0

3.1. Lower and upper estimate of solution. In this subsection, we study the asymptotic behavior of solutions (whenever exists) at origin of Eq.(1.36).

Following Lemma 3.1 and 3.2 are crucially used to prove Theorem 1.3.

Lemma 3.1. *Let $q < \frac{2+\nu}{\nu}$ and $\nu \in (0, \frac{N-2}{2})$. Then there exists $l > 0$ (can be chosen small) such that the following problem*

$$(3.1) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla w) + |x|^{-(q+1)\nu} w^q = 0 & \text{in } B_l(0) \\ w > 0 & \text{in } B_l(0) \\ w \in H^1(B_l(0), |x|^{-2\nu}) \cap L^{q+1}(B_l(0), |x|^{-(q+1)\nu}), \end{cases}$$

has a continuous radial solution w_1 such that $w_1(0) = 1$.

Proof. To prove this lemma, it is enough to show that the following ODE has a unique solution w_1 in $(0, l)$ for some $l > 0$ and w_1 is a solution of Eq.(3.1),

$$(3.2) \quad \begin{cases} w'' + \frac{N-1-2\nu}{r} w'(r) = r^{-(q-1)\nu} w^q & \text{in } (0, 1) \\ w > 0 & \text{in } (0, 1) \\ w(0) = 1 \end{cases}$$

We can write a solution of the above ODE as

$$(3.3) \quad w(r) = 1 + \int_0^r s^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} w^q(t) dt ds.$$

Since $q < \frac{2+\nu}{\nu}$, using Banach fixed point theorem, it is easy to check that solution of the integral equation (3.3) exists and unique in $(0, l)$ for some $l > 0$. From (3.3), it follows w is continuous in $[0, l]$ and

$$w'(r) = r^{2\nu+1-N} \int_0^r t^{N-1-(q+1)\nu} w^q(t) dt \quad \text{for } r > 0.$$

Therefore by a straight forward computation it follows

$$(3.4) \quad \int_0^l w'(r)^2 r^{N-1-2\nu} dr < \infty \quad \text{and} \quad \int_0^l w^{q+1}(r) r^{N-1-(q+1)\nu} < \infty$$

as $q < \frac{2+\nu}{\nu}$ and $\nu < \frac{N-2}{2}$. Define $w_1(x) := w(r)$, where $r = |x|$.

Claim: w_1 is a weak solution of Eq.(3.1).

Indeed by (3.4), $w_1 \in H^1(B_l(0), |x|^{-2\nu}) \cap L^{q+1}(B_l(0), |x|^{-(q+1)\nu})$. Choose $0 < \eta < l$ and define $\chi_\eta \in C_0^\infty(B_l(0))$ such that $\chi_\eta = 1$ for $|x| \leq \frac{\eta}{2}$, $\chi_\eta = 0$ for $|x| > \eta$ and $|\nabla \chi_\eta| \leq \frac{4}{\eta}$. Let $\phi \in C_0^\infty(B_l(0))$ be arbitrarily chosen. Set $D_\eta := B_l(0) \setminus B_{\frac{\eta}{2}}(0)$. Therefore,

$$\begin{aligned} & \int_{B_l(0)} |x|^{-2\nu} \nabla w_1 \nabla \phi dx + \int_{B_l(0)} |x|^{-(q+1)\nu} w_1^q \phi dx \\ &= \lim_{\eta \rightarrow 0} \int_{B_l(0)} \chi_\eta |x|^{-2\nu} \nabla w_1 \nabla \phi dx + \lim_{\eta \rightarrow 0} \int_{B_l(0)} \chi_\eta |x|^{-(q+1)\nu} w_1^q \phi dx \\ &+ \lim_{\eta \rightarrow 0} \int_{B_l(0)} (1 - \chi_\eta) (|x|^{-2\nu} \nabla w_1 \nabla \phi + |x|^{-(q+1)\nu} w_1^q \phi) dx \\ &= - \lim_{\eta \rightarrow 0} \int_{D_\eta} (\nabla(1 - \chi_\eta) \nabla w_1) |x|^{-2\nu} \phi \\ (3.5) \quad & - \lim_{\eta \rightarrow 0} \int_{D_\eta} (1 - \chi_\eta) \left(\operatorname{div}(|x|^{-2\nu} \nabla w_1) - |x|^{-(q+1)\nu} w_1^q \right) \phi dx \end{aligned}$$

Since w_1 is a solution of the ODE (3.2) in $(0, l)$ and it is C^1 away from 0, it easily follows that w_1 is a C^1 solution of Eq.(3.1) in D_η , for every $\eta > 0$. Thus the last integral in (3.5) equals 0. Furthermore,

$$\left| \lim_{\eta \rightarrow 0} \int_{D_\eta} \nabla(1 - \chi_\eta) \nabla |x|^{-2\nu} w_1 \phi \right| \leq \lim_{\eta \rightarrow 0} C \eta^N \cdot \eta^{-1-2\nu+2\nu+1-(q+1)\nu} = 0.$$

Hence (3.5) yields

$$\int_{B_l(0)} |x|^{-2\nu} \nabla w_1 \nabla \phi dx + \int_{B_l(0)} |x|^{-(q+1)\nu} w_1^q \phi dx = 0,$$

which in turn proves the claim. This completes the proof of the lemma. \square

Remark 3.1. It is easy to see that (3.3) is related to a 2nd order ODE and solving this ODE requires two initial/boundary conditions. In our case it is natural to have initial values on $u(0)$ and $u'(0)$. But it is not hard to see from (3.3) that $u'(0)$ is not defined for $\frac{1+\nu}{\nu} \leq q < \frac{2+\nu}{\nu}$. Therefore a standard ODE technique does not give

existence of solution here. Moreover, as the solution of the integral equation (3.3) is not differentiable at 0, it does not directly follow that w is a solution of the given PDE (3.1).

Lemma 3.2. *Let $m > 0$, $q < \frac{2+\nu}{\nu}$ and $\nu \in (0, \frac{N-2}{2})$. Then for some $\delta \in (0, 1)$, there exists a radial continuous solution w_δ of Eq. (3.1) in $B_l(0)$, where l is as in Lemma 3.1, with the property that $w_\delta(0) = \delta$ and $w_\delta|_{\partial B_l(0)} < m$ and $\int_0^l |w'_\delta(r)|^2 r^{N-1-2\nu} dr < \infty$.*

Proof. Given $\delta > 0$, let w_δ be the solution of

$$(3.6) \quad \begin{cases} u'' + \frac{N-1-2\nu}{r} u'(r) = r^{-(q-1)\nu} u^q & \text{in } (0, l_\delta), \\ u > 0 & \text{in } (0, l_\delta), \\ u(0) = \delta, \end{cases}$$

where $[0, l_\delta]$ is maximum neighbourhood of 0 where the solution exists. Due to local existence, we have $l_\delta > 0$ (see for instance, Lemma 3.1). Moreover, we can write the solution as

$$(3.7) \quad w_\delta(r) = \delta + \int_0^r s^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} w_\delta^q(t) dt ds.$$

Claim 1. : If $0 < \delta_1 < \delta_2 \leq 1$, then $w_{\delta_1} \leq w_{\delta_2} \leq w_1$ in $[0, l]$, where w_1 and l are as in Lemma 3.1.

To see the claim, let $0 < \delta_1 < \delta_2 \leq 1$. Since $w_{\delta_1}(0) < w_{\delta_2}(0)$, there exists $r_0 > 0$ such that $w_{\delta_1} < w_{\delta_2}$ in $[0, r_0]$. Define

$$S := \{s \in [0, l] : w_{\delta_1}(s) > w_{\delta_2}(s)\}.$$

If $S = \emptyset$, then we are done. Suppose $S \neq \emptyset$. We define

$$\tilde{r}_0 := \inf S.$$

Clearly $\tilde{r}_0 > 0$. We show that $\tilde{r}_0 \not\leq l$. Indeed, from (3.7), we have

$$w'_{\delta_1}(r) - w'_{\delta_2}(r) = r^{2\nu+1-N} \int_0^r t^{N-1-(q-1)\nu} [w_{\delta_1}^q(t) - w_{\delta_2}^q(t)] dt.$$

Therefore $(w_{\delta_1} - w_{\delta_2})'(r) < 0$ for $r \in [0, \tilde{r}_0]$. This implies $w_{\delta_1}(\tilde{r}_0) < w_{\delta_2}(\tilde{r}_0)$, which is a contradiction to the definition of \tilde{r}_0 . Hence the claim follows.

Claim 2. : $w_\delta \rightarrow 0$ uniformly in $[0, l]$, as $\delta \rightarrow 0$.

Note that $\lim_{\delta \downarrow 0} w_\delta$ exists, since $w_\delta > 0$ and Claim 1 holds. Let $w := \lim_{\delta \downarrow 0} w_\delta$. Using monotone convergence theorem, we pass the limit in (3.7) to obtain

$$w(r) = \int_0^r s^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} w^q(t) dt ds.$$

Solution of this integral equation uniquely exists in $(0, l)$ (see for instance Lemma 3.1). Therefore $w = 0$. Hence the claim follows by Dini's theorem.

Combining Claim 1 and Claim 2, the lemma follows. \square

Proof of Theorem 1.3. Define, $v = |x|^\nu u$. Then it follows from [13, Theorem 1.1] that $v \in H_0^1(\Omega, |x|^{-2\nu})$ and v satisfies the following equation:

$$(3.8) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla v) = & |x|^{-(p+1)\nu} v^p - |x|^{-(q+1)\nu} v^q & \text{in } \Omega, \\ v > & 0 & \text{in } \Omega, \\ v \in & H_0^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu}). \end{cases}$$

By elliptic regularity theory $v \in C^2(\Omega \setminus \{0\}) \cap C^1(\bar{\Omega} \setminus \{0\})$ (see [12], [13]). It is easy to see that v is a super-solution of the following problem

$$(3.9) \quad \begin{aligned} -\operatorname{div}(|x|^{-2\nu} \nabla w) + |x|^{-(q+1)\nu} w^q &= 0 & \text{in } B_l(0), \\ w &= m & \text{on } \partial B_l(0), \\ w &> 0 & \text{in } B_l(0), \\ w &\in H^1(B_l(0), |x|^{-2\nu}) \cap L^{q+1}(B_l(0), |x|^{-(q+1)\nu}), \end{aligned}$$

where $l > 0$ is as in Lemma 3.1 and $0 < m < m_l = \min_{|x|=l} v$.

Claim: If w is any solution of (3.9), then $v \geq w$ in $B_l(0)$.

To see the claim, we note that $(v - w)$ satisfies

$$-\operatorname{div}(|x|^{-2\nu} \nabla (v - w)) \geq -|x|^{-(q+1)\nu} A(x)(v - w) \quad \text{in } B_l(0),$$

where $0 \leq A(x) := \frac{v^q(x) - w^q(x)}{v(x) - w(x)} \leq q \max[v(x), w(x)]^{q-1}$. Moreover, $w \leq v$ on $\partial B_l(0)$. Thus taking $(v - w)^-$ as the test function we obtain

$$\int_{B_l(0)} |x|^{-2\nu} |\nabla (v - w)|^2 dx + \int_{B_l(0)} |x|^{-(q+1)\nu} A(x) |(v - w)^-|^2 dx \leq 0,$$

which implies $v \geq w$ in $B_l(0)$.

By Lemma 3.2, it follows that Eq. (3.9) admits a solution w_δ with $w_\delta(0) = \delta > 0$. As a result $\lim_{|x| \rightarrow 0} v(x) \geq \delta$, which in turn implies

$$u(x) \geq c|x|^{-\nu}, \quad x \in B_{r_0}(0) \setminus \{0\},$$

for some $r_0 > 0$ small. □

Proof of Theorem 1.4. We prove this theorem in the spirit of [13].

Define,

$$(3.10) \quad v(x) = |x|^\nu u(x) \quad \text{and} \quad f(x, u) = u^p - u^q.$$

Then Eq.(1.36) reduces to

$$(3.11) \quad -\operatorname{div}(|x|^{-2\nu} \nabla v) = |x|^{-(p+1)\nu} v^p - |x|^{-(q+1)\nu} v^q \quad \forall x \in \Omega \setminus \{0\}.$$

By elliptic regularity theory $v \in C^2(\Omega \setminus \{0\}) \cap C^1(\bar{\Omega} \setminus \{0\})$ (see [12], [13]). Let $\rho > 0$ small enough such that $B_\rho(0) \Subset \Omega$. For $s, l > 1$, we choose the test function φ as follows:

$$\begin{aligned} \varphi &= \eta^2 v v_l^{2(s-1)} \in H_0^1(\Omega, |x|^{-2\nu} dx), \\ v_l &= \min\{v, l\}, \quad \eta \in C_0^\infty(B_\rho(0)), \end{aligned}$$

with the properties $0 \leq \eta \leq 1$, $\eta = 1$ in $B_r(0)$, $r < \rho$ and $|\nabla \eta| \leq \frac{4}{\rho-r}$. Using this test function φ , we obtain from (3.11),

$$(3.12) \quad \int_{\Omega} |x|^{-2\nu} \nabla v \nabla \varphi dx = \int_{\Omega} (|x|^{-(p+1)\nu} v^p - |x|^{-(q+1)\nu} v^q) \varphi dx.$$

Substituting the function f , RHS of (3.12) can be simplified as below

$$(3.13) \quad RHS = \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^{p+1} v_l^{2(s-1)} dx - \int_{\Omega} |x|^{-(q+1)\nu} \eta^2 v^{q+1} v_l^{2(s-1)} dx.$$

After doing a standard computation, the LHS of (3.12) can be rewritten as:

$$(3.14) \quad \int_{\Omega} |x|^{-2\nu} \times \left(2\eta v v_l^{2(s-1)} \nabla \eta \nabla v + \eta^2 v_l^{2(s-1)} |\nabla v|^2 + 2(s-1) \eta^2 v_l^{2(s-1)} |\nabla v_l|^2 \right) dx.$$

Using Cauchy-Schwartz inequality, for any $\epsilon > 0$ we have,

$$(3.15) \quad \begin{aligned} \left| 2 \int_{\Omega} |x|^{-2\nu} \eta v v_l^{2(s-1)} \nabla \eta \nabla v dx \right| &\leq \epsilon \int_{\Omega} |x|^{-2\nu} \eta^2 v_l^{2(s-1)} |\nabla v|^2 dx \\ &+ C(\epsilon) \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 |v|^2 v_l^{2(s-1)} dx. \end{aligned}$$

Combining (3.12)–(3.15) we obtain,

$$(3.16) \quad \begin{aligned} &\int_{\Omega} |x|^{-2\nu} \left(\eta^2 v_l^{2(s-1)} |\nabla v|^2 + 2(s-1) \eta^2 v_l^{2(s-1)} |\nabla v_l|^2 \right) dx \\ &\leq C \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 v^2 v_l^{2(s-1)} dx \\ &+ \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^{p+1} v_l^{2(s-1)} dx \\ &- \int_{\Omega} |x|^{-(q+1)\nu} \eta^2 v^{q+1} v_l^{2(s-1)} dx. \end{aligned}$$

We recall here Caffarelli-Kohn-Nirenberg inequality (see [3]):

$$(3.17) \quad \left(\int_{\Omega} |x|^{-br} |w|^r dx \right)^{\frac{2}{r}} \leq C_{a,b} \int_{\Omega} |x|^{-2a} |\nabla w|^2 dx \quad \forall \quad w \in H_0^1(\Omega, |x|^{-2a} dx),$$

where $-\infty < a < \frac{N-2}{2}$, $a \leq b \leq a+1$, $r = \frac{2N}{N-2+2(b-a)}$ and $C_{a,b}$ is a positive constant.

Let $w = \eta v v_l^{s-1}$ and $a = b = \nu < \frac{N-2}{2}$ in (3.17). Then $r = 2^*$. Consequently we get from (3.17),

$$(3.18) \quad \left(\int_{\Omega} |x|^{-2^*\nu} |\eta v v_l^{s-1}|^{2^*} dx \right)^{\frac{2}{2^*}} \leq C_{a,a} \int_{\Omega} |x|^{-2\nu} |\nabla(\eta v v_l^{s-1})|^2 dx.$$

Using (3.16), we simplify the RHS of (3.18) as in [13], i.e.,

$$(3.19) \quad \begin{aligned} RHS &\leq 2C_{a,a} \int_{\Omega} |x|^{-2\nu} \\ &\times \left(|\nabla \eta|^2 v^2 v_l^{2(s-1)} + \eta^2 v_l^{2(s-1)} |\nabla v|^2 + (s-1)^2 \eta^2 v_l^{2(s-1)} |\nabla v_l|^2 \right) dx \\ &\leq Cs \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 v^2 v_l^{2(s-1)} + Cs \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^{p+1} v_l^{2(s-1)} dx \\ &- \int_{\Omega} |x|^{-(q+1)\nu} \eta^2 v^{q+1} v_l^{2(s-1)} dx \\ &\leq Cs \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 v^2 v_l^{2(s-1)} + Cs \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^{p+1} v_l^{2(s-1)} dx. \end{aligned}$$

For $p \geq 2^* - 1$, choose $t > 1$ as follows:

$$(3.20) \quad \frac{N}{2} < t < \frac{q+1}{p-1}.$$

Note that for $p = 2^* - 1$ the interval $(\frac{N}{2}, \frac{q+1}{p-1})$ is always a nonempty set. On the other hand, $(\frac{N}{2}, \frac{q+1}{p-1}) \neq \emptyset$, since $q > (p-1)\frac{N}{2} - 1$. From (3.20) we have,

$$(p-1)t < q+1 \quad \text{and} \quad 2 < \frac{2t}{t-1} < 2^*.$$

Consequently

$$(3.21) \quad \begin{aligned} \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^{p+1} v_l^{2(s-1)} dx &= \int_{\Omega} \eta^2 u^{p+1} v_l^{2(s-1)} dx \\ &= \int_{\Omega} |\eta v v_l^{s-1}|^2 u^{p-1} |x|^{-2\nu} dx \\ &\leq \|u\|_{L^{(p-1)t}(\Omega)}^{p-1} \| |x|^{-\nu} \eta v v_l^{s-1} \|_{L^{\frac{2t}{t-1}}(\Omega)}^2 \\ &\leq C(\epsilon \| |x|^{-\nu} \eta v v_l^{s-1} \|_{L^{2^*}(\Omega)} \\ &\quad + C(N, t) \epsilon^{-\frac{N}{2t-N}} \| |x|^{-\nu} \eta v v_l^{s-1} \|_{L^2(\Omega)})^2 \\ &\leq C\epsilon^2 \left(\int_{\Omega} |x|^{-2^*\nu} |\eta v v_l^{s-1}|^{2^*} dx \right)^{\frac{2}{2^*}} \\ &\quad + C\epsilon^{-\frac{2N}{2t-N}} \int_{\Omega} |x|^{-2\nu} |\eta v v_l^{s-1}|^2 dx. \end{aligned}$$

Plugging (3.21) into (3.19) and then (3.19) into (3.18), we have

$$(3.22) \quad \begin{aligned} \left(\int_{\Omega} |x|^{-2^*\nu} |\eta v v_l^{s-1}|^{2^*} dx \right)^{\frac{2}{2^*}} &\leq Cs \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 v^2 v_l^{2(s-1)} dx \\ &\quad + Cs\epsilon^2 \left(\int_{\Omega} |x|^{-2^*\nu} |\eta v v_l^{s-1}|^{2^*} dx \right)^{\frac{2}{2^*}} \\ &\quad + Cs\epsilon^{-\frac{2N}{2t-N}} \int_{\Omega} |x|^{-2\nu} |\eta v v_l^{s-1}|^2 dx. \end{aligned}$$

By choosing $\epsilon = \frac{1}{\sqrt{2Cs}}$, we obtain from (3.22)

$$(3.23) \quad \begin{aligned} \left(\int_{\Omega} |x|^{-2^*\nu} |\eta v v_l^{s-1}|^{2^*} dx \right)^{\frac{2}{2^*}} &\leq Cs \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 v^2 v_l^{2(s-1)} dx \\ &\quad + Cs^{\frac{2t}{2t-N}} \int_{\Omega} |x|^{-2\nu} |\eta v v_l^{s-1}|^2 dx \\ &\leq Cs^{\alpha} \int_{\Omega} |x|^{-2\nu} (\eta^2 + |\nabla \eta|^2) v^2 v_l^{2(s-1)} dx, \end{aligned}$$

where $\alpha = \frac{2t}{2t-N}$. Moreover, it is not difficult to check that

$$\int_{\Omega} |x|^{-2^*\nu} |\eta v v_l^{s-1}|^{2^*} dx \geq \int_{\Omega} |x|^{-2^*\nu} |\eta|^{2^*} v^2 v_l^{2^*s-2} dx.$$

Consequently as in [13], we have

$$\begin{aligned}
 \left(\int_{\Omega} |x|^{-2^* \nu} |\eta|^{2^*} v^2 v_l^{2^* s-2} dx \right)^{\frac{2}{2^*}} &\leq C s^{\alpha} \int_{\Omega} |x|^{-2\nu} (\eta^2 + |\nabla \eta|^2) v^2 v_l^{2(s-1)} dx \\
 (3.24) \qquad \qquad \qquad &\leq C s^{\alpha} \int_{\Omega} |x|^{-2^* \nu} (\eta^2 + |\nabla \eta|^2) v^2 v_l^{2(s-1)} dx.
 \end{aligned}$$

Substituting η and $\nabla \eta$ we deduce

$$(3.25) \quad \left(\int_{B_r(0)} |x|^{-2^* \nu} v^2 v_l^{2^* s-2} dx \right)^{\frac{2}{2^*}} \leq \frac{C s^{\alpha}}{(\rho - r)^2} \int_{B_{\rho}(0)} |x|^{-2^* \nu} v^2 v_l^{2s-2} dx.$$

Set s^* and s_j as follows:

$$\frac{N}{N-2} < s^* < \frac{q+1}{2} \quad \text{and} \quad s_j = s^* \left(\frac{2^*}{2} \right)^j, \quad j = 1, 2, \dots.$$

If we take $s = s_j$ in (3.25), a straight forward computation yields:

$$(3.26) \quad \left(\int_{B_r(0)} |x|^{-2^* \nu} v^2 v_l^{2s_{j+1}-2} dx \right)^{\frac{1}{2s_{j+1}}} \leq \left(\frac{C s^{\alpha}}{(\rho - r)^2} \right)^{\frac{1}{2s_j}} \left(\int_{B_{\rho}(0)} |x|^{-2^* \nu} v^2 v_l^{2s_j-2} dx \right)^{\frac{1}{2s_j}}.$$

Choose $\rho_0 > 0$ such that $B_{2\rho_0} \Subset \Omega$ and $r_j = \rho_0(1 + \rho_0^j)$, $j = 1, 2, \dots$. By taking $\rho = r_j$, $r = r_{j+1}$ in (3.26) and following the calculation of [13] we find:

$$\begin{aligned}
 \left(\int_{B_{r_{j+1}}(0)} |x|^{-2^* \nu} v^2 v_l^{2s_{j+1}-2} dx \right)^{\frac{1}{2s_{j+1}}} &\leq \left(\frac{C}{(1 - \rho_0)\rho_0} \right)^{\sum_{j=0}^{\infty} \frac{1}{2s_j} - \sum_{j=0}^{\infty} \frac{j}{2s_j}} \times \\
 (3.27) \qquad \qquad \qquad &\prod_{j=0}^{\infty} s_j^{\frac{\alpha}{2s_j}} \left(\int_{B_{r_0}(0)} |x|^{-2^* \nu} v^2 v_l^{2s^*-2} dx \right)^{\frac{1}{2s^*}}.
 \end{aligned}$$

By standard computation it follows that (see [13])

$$\sum_{j=0}^{\infty} \frac{1}{2s_j} \leq C, \quad \sum_{j=0}^{\infty} \frac{j}{2s_j} \leq C \quad \text{and} \quad \prod_{j=0}^{\infty} s_j^{\frac{\alpha}{2s_j}} \leq C.$$

Since $2^* < 2s^* < q+1$, after a straight forward computation as in [13], we obtain

$$\int_{B_{r_0}(0)} |x|^{-2^* \nu} v^2 v_l^{2s^*-2} dx \leq (\text{diam } \Omega)^{(2s^*-2^*)\nu} \int_{\Omega} u^{2s^*} dx \leq C.$$

As a result, from (3.27) we have

$$(3.28) \quad \left(\int_{B_{r_{j+1}}(0)} |x|^{-2^* \nu} v^2 v_l^{2s_{j+1}-2} dx \right)^{\frac{1}{2s_{j+1}}} \leq C.$$

Moreover,

$$\begin{aligned}
 \text{LHS of (3.28)} &\geq \left(\int_{B_{r_{j+1}}(0)} |x|^{-2^* \nu} v_l^{2s_{j+1}} dx \right)^{\frac{1}{2s_{j+1}}} \\
 (3.29) \qquad \qquad &\geq (\text{diam } \Omega)^{\frac{-2^* \nu}{2s_{j+1}}} |v_l|_{L^{2s_{j+1}}(B_{\rho_0}(0))}
 \end{aligned}$$

Combining (3.29) with (3.28), we obtain

$$|v_l|_{L^{2s_{j+1}}(B_{\rho_0}(0))} \leq C(\text{diam } \Omega)^{\frac{2^*\nu}{2s_{j+1}}}.$$

Note that $s_{j+1} \rightarrow \infty$ as $j \rightarrow \infty$. Hence $|v_l|_{L^\infty(B_{\rho_0}(0))} \leq C$. Finally letting $l \rightarrow \infty$ we have $|v|_{L^\infty(B_{\rho_0}(0))} \leq C$, which in turn implies

$$u(x) \leq C|x|^{-\nu} \quad \forall \quad x \in B_{\rho_0}(0) \setminus \{0\}.$$

□

Proof of Theorem 1.5. We use an idea from [11]. If u is a positive solution of Eq. (1.36), then u satisfies

$$-\Delta u - \mu \frac{u}{|x|^2} = -(1 + o(1))u^q, \quad \text{in } B_R(0),$$

for some $R > 0$ small. Using the transformation $v = |x|^\nu u$, we get v satisfies

$$(3.30) \quad -\text{div}(|x|^{-2\nu} \nabla v) = -(1 + o(1))|x|^{-(q+1)\nu} v^q, \quad \text{in } B_R(0).$$

Therefore we can write

$$(3.31) \quad -\text{div}(|x|^{-2\nu} \nabla v) = -A|x|^{-(q+1)\nu} v^q, \quad \text{in } B_R(0),$$

where $1 - \delta < A < 1 + \delta$, for some $\delta > 0$.

Claim: $v(x) \leq C|x|^{\nu - \frac{2}{q-1}}$ in $B_{\frac{2R}{3}}(0) \setminus \{0\}$, for some $C = C(N, q, p, R, \mu) > 0$.

To see the claim, for $0 < r < R$, set

$$y = \frac{x}{r} \quad \text{and} \quad w(y) = r^{-\nu + \frac{2}{q-1}} v(x).$$

Then w satisfies Eq. (3.31) in $B_1(0)$.

Now define

$$W(y) := c \left[\left(\frac{9}{16} - |y|^2 \right) \left(|y|^2 - \frac{1}{16} \right) \right]^{-\beta},$$

where $\beta > \frac{2}{q-1}$ and $c > 0$ will be chosen later. Clearly

$$W = \infty \quad \text{on} \quad \partial(B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0)).$$

We show that β and c in the definition of W can be chosen such that

$$-\text{div}(|x|^{-2\nu} \nabla W) \geq -A|x|^{-(q+1)\nu} W^q, \quad \text{in } B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0).$$

Since W is radial, it is enough to show that

$$(3.32) \quad W'' + \frac{N-1-2\nu}{r} W' \leq A r^{-(q-1)\nu} W^q, \quad \frac{1}{4} < r < \frac{3}{4}.$$

By a direct computation, when $\frac{1}{4} < r < \frac{3}{4}$, we obtain

$$\begin{aligned}
W'' + \frac{N-1-2\nu}{r}W' &= -2W\beta\left[-\left(\frac{9}{16}-r^2\right)^{-1} + \left(r^2-\frac{1}{16}\right)^{-1}\right](N+2r^2-2\nu) \\
&\quad + 4r^2W\beta\left[\left(\frac{9}{16}-r^2\right)^{-2} + \left(r^2-\frac{1}{16}\right)^{-2}\right] \\
&\leq CW\beta\left[\left(\frac{9}{16}-r^2\right)^{-2} + \left(r^2-\frac{1}{16}\right)^{-2}\right] \\
&\leq CW\beta\left[\left(\frac{9}{16}-r^2\right)^{-2}\left(r^2-\frac{1}{16}\right)^{-2}\right] \\
&= C\beta W^{1+\frac{2}{\beta}}
\end{aligned}$$

Since $\beta > \frac{2}{q-1}$ implies $1 + \frac{2}{\beta} < q$, (3.32) follows. Therefore we obtain,

$$-\operatorname{div}(|x|^{-2\nu}\nabla(W-w)) \geq -A|x|^{-(q+1)\nu}B(x)(W-w) \quad \text{in } B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0),$$

where $0 \leq B(x) := \frac{W^q(x)-w^q(x)}{W(x)-w(x)} \leq q \max[W(x), w(x)]^{q-1}$. Moreover, $(W-w)^- = 0$ on $\partial(B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0))$. Thus taking $(W-w)^-$ as the test function we obtain

$$\int_{B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0)} |x|^{-2\nu} |\nabla(W-w)^-|^2 dx + \int_{B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0)} A|x|^{-(q+1)\nu} B(x) |(W-w)^-|^2 dx \leq 0,$$

which implies $w \leq W$ in $B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0)$.

In particular,

$$w(y) \leq \max_{\frac{1}{3} < |y| < \frac{2}{3}} W(y) \quad \text{in } B_{\frac{2}{3}}(0) \setminus B_{\frac{1}{3}}(0),$$

which yields

$$\max_{\frac{r}{3} < |x| < \frac{2r}{3}} v(x) \leq C|x|^{\nu-\frac{2}{q-1}}.$$

Since $0 < r < R$ was arbitrarily chosen, the claim follows.

Hence $u(x) \leq C|x|^{-\frac{2}{q-1}}$ in $B_{\frac{2R}{3}}(0) \setminus \{0\}$. From Theorem 1.4, it also follows that $u(x) \leq C|x|^{-\nu}$. Since $q > \frac{2+\nu}{\nu}$ implies $|x|^{-\frac{2}{q-1}} \leq C|x|^{-\nu}$, the theorem follows. \square

Proof of Theorem 1.6. If u is a positive solution of Eq. (1.36), then as in the proof of Theorem 1.5, u satisfies

$$-\Delta u - \mu \frac{u}{|x|^2} = -(1+o(1))u^q, \quad \text{in } B_R(0),$$

for some $R > 0$ small. Using the transformation $v = |x|^\nu u$, we get v satisfies

$$-\operatorname{div}(|x|^{-2\nu}\nabla v) = -(1+o(1))|x|^{-(q+1)\nu}v^q, \quad \text{in } B_R(0).$$

Given $\delta > 0$, we can write

$$(3.33) \quad -\operatorname{div}(|x|^{-2\nu}\nabla v) = -A|x|^{-(q+1)\nu}v^q, \quad \text{in } B_R(0),$$

where $1-\delta < A < 1+\delta$. Define

$$(3.34) \quad V(x) := c|x|^{\nu-\frac{2}{q-1}},$$

where $c < \min\{c_1, c_2\}$,

$$c_1 := R^{-\nu+\frac{2}{q-1}} \min_{|x|=R} v,$$

and c_2 is defined in (3.40). Therefore, it is easy to see

$$(3.35) \quad v \geq V \quad \text{on} \quad \partial B_R(0).$$

Claim: $-\operatorname{div}(|x|^{-2\nu}\nabla V) \leq -A|x|^{-(q+1)\nu}V^q$ in $B_R(0)$.

To prove the claim, we note that since V is radial, it is enough to show that

$$V'' + \frac{N-1-2\nu}{r}V' \geq Ar^{-(q-1)\nu}V^q \quad r \in (0, R).$$

Using the Emden-Fowler transformation

$$(3.36) \quad y(t) = \alpha^\nu V(r), \quad t = \left(\frac{\alpha}{r}\right)^\alpha,$$

where $\alpha = N - 2 - 2\nu$, it is equivalent to prove that

$$(3.37) \quad y''(t) \geq At^{\frac{-(2\alpha+2)+(q-1)\nu}{\alpha}}y^q(t), \quad t > \left(\frac{\alpha}{R}\right)^\alpha.$$

Using (3.34) in (3.36), it is not difficult to see that $y(t) = c\alpha^{-\frac{2}{q-1}}t^{-\frac{1}{\alpha}(\nu-\frac{2}{q-1})}$. Consequently, by a straight forward computation we obtain,

$$(3.38) \quad y''(t) = c\alpha^{-\frac{q+1}{q-1}}\left(\nu - \frac{2}{q-1}\right)\left[\left(\nu - \frac{2}{q-1}\right)\frac{1}{\alpha} + 1\right]t^{-\left(\nu-\frac{2}{q-1}\right)\frac{1}{\alpha}-2}.$$

On the other hand, by direct computation it follows

$$(3.39) \quad At^{\frac{-(2\alpha+2)+(q-1)\nu}{\alpha}}y^q(t) = A(c\alpha^{-\frac{2}{q-1}})^qt^{-\left(\nu-\frac{2}{q-1}\right)\frac{1}{\alpha}-2}.$$

Define

$$(3.40) \quad c_2 := \left(\frac{1}{A}\alpha\left(\nu - \frac{2}{q-1}\right)\left[\left(\nu - \frac{2}{q-1}\right)\frac{1}{\alpha} + 1\right]\right)^{\frac{1}{q-1}}.$$

Note that $q > \frac{2+\nu}{\nu}$ implies $\nu - \frac{2}{q-1} > 0$. Therefore, since $c < \min\{c_1, c_2\}$, comparing (3.38) and (3.39), we conclude (3.37) holds true. Hence the claim follows.

We also note that both v and V are bounded in $B_R(0)$ (for V it follows from Theorem 1.5). Therefore combining the Claim and (3.35) and using comparison principle as in previous theorem, we obtain $v \geq V$ in $B_R(0)$. This in turn implies, $u(x) \geq c|x|^{-\frac{2}{q-1}}$ in $B_R(0) \setminus \{0\}$ which completes the proof. \square

3.2. The Critical Case $q = \frac{2+\nu}{\nu}$.

Proof of Theorem 1.7. Let u be any radial solution of Eq.(1.36) with $q = \frac{2+\nu}{\nu}$. Note that this implies $\nu = \frac{2}{q-1}$. Then as in the proof of previous theorem, $v = r^\nu u$ satisfies

$$(3.41) \quad v'' + \frac{N-1-2\nu}{r}v' = Ar^{-2}v^q, \quad r \geq R,$$

where $1 - \delta < A < 1 + \delta$, for some $\delta > 0$ (see (3.33)). Using the Emden-Fowler transformation

$$(3.42) \quad y(t) = \alpha^\nu v(r), \quad t = \left(\frac{\alpha}{r}\right)^\alpha,$$

where $\alpha = N - 2 - 2\nu$, (3.41) reduces to

$$(3.43) \quad t^2y''(t) - Ay^q(t) = 0, \quad t > \left(\frac{\alpha}{R}\right)^\alpha.$$

Claim: $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

To see the claim, we note that for large t , y' is increasing and nonnegative. From Theorem 1.4, we have v is bounded near 0. Therefore y is bounded near infinity. Using this fact, it is easy to check that $\lim_{t \rightarrow \infty} y'(t) = 0$. Consequently, $y'(t) \leq 0$ for large t which implies y is decreasing for large t . Hence $\lim_{t \rightarrow \infty} y(t) = c < +\infty$. If $c \neq 0$

$$y'(\theta) = - \int_{\theta}^{\infty} \frac{y^q}{s^2} ds, \text{ implies } y(T) = y(t) + \int_T^t \int_{\theta}^{\infty} \frac{y^q}{s^2} ds d\theta.$$

Hence we have

$$y(T) \geq y(t) + \left(\frac{c}{2}\right)^q \log \frac{t}{T}.$$

Since y is bounded, taking the limit $t \rightarrow \infty$ in the above expression yields a contradiction. Hence $c = 0$ and the claim follows.

Setting

$$(3.44) \quad t = e^s \quad \text{and} \quad x(s) = y(t),$$

(3.43) yields

$$(3.45) \quad x''(s) - x'(s) - Ax^q(s) = 0 \quad s \geq R',$$

where $R' = \log \frac{\alpha}{R}$. We are only interested in the solutions of (3.45), $x(s) \rightarrow 0$ as $s \rightarrow \infty$. Following an argument along the same line of [25, Lemma 3.2], it can be shown that

$$(3.46) \quad x(s) = \left(\frac{1}{(q-1)s} \right)^{\frac{1}{q-1}} \left(1 + \frac{q}{(q-1)^2} \frac{\log s}{s} (1 + o(1)) \right).$$

Using (3.42) and (3.44) and the fact that $\nu = \frac{2}{q-1}$, we obtain

$$(3.47) \quad v(r) = \left(\frac{\alpha}{q-1} \right)^{\frac{\nu}{2}} \left(\log \frac{\alpha}{r} \right)^{-\frac{\nu}{2}} \left[1 + \frac{q}{(q-1)^2} \frac{\log(\alpha \log \frac{\alpha}{r})}{\alpha \log \frac{\alpha}{r}} (1 + o(1)) \right].$$

Therefore

$$u(r) = \left(\frac{\alpha}{q-1} \right)^{\frac{\nu}{2}} r^{-\nu} (-\log r)^{-\frac{\nu}{2}} \left(1 - \frac{\log \alpha}{\log r} \right)^{-\frac{\nu}{2}} \left[1 + \frac{q}{(q-1)^2} \frac{\log(\alpha \log \frac{\alpha}{r})}{\alpha \log \frac{\alpha}{r}} (1 + o(1)) \right].$$

Hence it is easy to see that

$$\lim_{|x| \rightarrow 0} \frac{u(x)}{|x|^{-\nu} |\log |x||^{-\frac{\nu}{2}}} = \left(\frac{\alpha \nu}{2} \right)^{\frac{\nu}{2}}.$$

□

3.3. Gradient estimate. In this subsection we establish gradient estimate of any solution of Eq. (1.36) near origin. More precise we prove Theorem 1.8. Towards this goal, first we need the following two lemmas.

Lemma 3.3. *Let ρ_0, ρ be as in Theorem 1.4 and Theorem 1.5, respectively, u be a weak solution of Eq. (1.36) and p, q be as in Theorem 1.8. Then there exists $\mu^* = \mu^*(N, q) > 0$ and a constant C depending on N, p, q, μ such that u satisfies*

$$\int_{B_{\frac{|x|}{4}}(x)} |\nabla u(x)|^2 dx \leq \begin{cases} C|x|^{-2(\nu+1)} & \text{if } \mu \in [0, \mu^*), \\ C|x|^{-2(\frac{q+1}{4-1})} & \text{if } \mu \in [\mu^*, \bar{\mu}), \end{cases}$$

for $0 < |x| < \frac{1}{2} \min\{\rho_0, \rho\}$.

Proof. Define $\tilde{\rho}_0 := \frac{1}{2} \min\{\rho_0, \rho\}$. Fix $x \in \mathbb{R}^N$ such that $0 < |x| < \tilde{\rho}_0$. Let $B = B_{\frac{|x|}{4}}(x)$ and $2B = B_{\frac{|x|}{2}}(x)$. Choose $\eta \in C_0^\infty(2B)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B and $|\nabla \eta| < \frac{8}{|x|}$. Define $\varphi := \eta^2 u$. Using this test function φ , we obtain from Eq. (1.36)

$$\int_{2B} \nabla u \nabla \varphi dx = \int_{2B} \left(\frac{\mu u^2 \eta^2}{|x|^2} + u^{p+1} \eta^2 - u^{q+1} \eta^2 \right) dx.$$

Moreover, by a straight forward computation it follows

$$\int_{2B} \nabla u \nabla \varphi dx \geq \frac{1}{2} \int_B |\nabla u|^2 dx - C \int_{2B} u^2 |\nabla \eta|^2 dx.$$

Therefore we have

$$(3.48) \quad \int_B |\nabla u|^2 dx \leq C \int_{2B} \left(u^2 |\nabla \eta|^2 + \frac{\mu u^2 \eta^2}{|x|^2} + u^{p+1} \eta^2 - u^{q+1} \eta^2 \right) dx.$$

Define

$$(3.49) \quad \mu^* = \left(\frac{N-2}{2} \right)^2 - \left(\frac{N-2}{2} - \frac{2}{q-1} \right)^2.$$

We observe that $\mu < \mu^* \iff q < \frac{2+\nu}{\nu}$.

Case 1: $q < \frac{2+\nu}{\nu}$.

Applying Theorem 1.4 in (3.48), we obtain

$$(3.50) \quad \int_B |\nabla u|^2 dx \leq C \left(|x|^{-2\nu+N} + |x|^{-2\nu-2+N} + |x|^{-(p+1)\nu+N} + |x|^{-(q+1)\nu+N} \right),$$

for every x satisfying $0 < |x| < \tilde{\rho}_0$. Therefore from (3.50), we have

$$(3.51) \quad \int_B |\nabla u|^2 dx \leq C |x|^{-(2\nu+2)+N} \quad \text{if } \mu \in (0, \mu^*).$$

Case 2: $q \geq \frac{2+\nu}{\nu}$.

In this case we have $\mu \geq \mu^*$. Applying Theorem 1.5 in (3.48), we obtain

$$(3.52) \quad \begin{aligned} \int_B |\nabla u|^2 dx &\leq C \left(|x|^{-\frac{4}{q-1}+N} + |x|^{-2(\frac{q+1}{q-1})+N} + |x|^{-2(\frac{p+1}{q-1})+N} + |x|^{-2(\frac{q+1}{q-1})+N} \right) \\ &\leq C |x|^{-2(\frac{q+1}{q-1})+N}, \end{aligned}$$

for $0 < |x| < \tilde{\rho}$.

Combining (3.51) and (3.52), the lemma follows. \square

The next lemma is due to Xiang, see [26, Proposition 2.1].

Lemma 3.4. *Let Ω be a domain in \mathbb{R}^N , $f \in L_{loc}^\infty(\Omega)$ and $u \in H_{loc}^1(\Omega)$ be a weak solution of the equation*

$$-\Delta u = f \quad \text{in } \Omega.$$

Then for any $B_{2R}(x_0) \subseteq \Omega$, it holds

$$\sup_{B_{\frac{R}{2}}(x_0)} |\nabla u| \leq C \left(\int_{B_R(x_0)} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} + CR |f|_{L^\infty(B_R(x_0))}.$$

Proof of Theorem 1.8. Let u be a weak solution of (1.36) and $\tilde{\rho}$ be as in Lemma 3.3. Then we can write $-\Delta u = f(x)$, where

$$f(x) = \mu \frac{u}{|x|^2} + u^p - u^q.$$

Case 1: $q < \frac{2+\nu}{\nu}$.

In this case by Theorem 1.4, it follows $|f(x)| \leq C(|x|^{-\nu-2} + |x|^{-\nu p} + |x|^{-\nu q})$. Since $q < \frac{2+\nu}{\nu} \iff \mu < \mu^*$, we have

$$(3.53) \quad |f(x)| \leq C|x|^{-\nu-2} \quad \text{if } \mu \in [0, \mu^*),$$

for $0 < |x| < \tilde{\rho}$.

Case 2: $q \geq \frac{2+\nu}{\nu}$.

In this case By Theorem 1.5, we obtain

$$(3.54) \quad |f(x)| \leq C(|x|^{-\frac{2q}{q-1}} + |x|^{-\frac{2p}{q-1}}) \leq C|x|^{-\frac{2q}{q-1}}, \quad \mu \in [\mu^*, \bar{\mu})$$

for $0 < |x| < \tilde{\rho}$.

Consequently, in both Case 1 and Case 2, $f \in L_{loc}^\infty(B_{\rho_0}(0) \setminus \{0\})$. As a result, for any $x \in B_{\tilde{\rho}}(0) \setminus \{0\}$, we apply Lemma 3.4 on the domain $B_{\frac{|x|}{2}}(x)$ to obtain that

$$\sup_{B_{\frac{|x|}{8}}(x)} |\nabla u| \leq C \left(\int_{B_{\frac{|x|}{4}}(x)} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} + C|x||f|_{L^\infty(B_{\frac{|x|}{4}}(x))}.$$

Combining (3.53), (3.54) and Lemma 3.3, we obtain from the above expression

$$\sup_{B_{\frac{|x|}{8}}(x)} |\nabla u| \leq \begin{cases} C|x|^{-(\nu+1)} & \text{if } \mu \in [0, \mu^*), \\ C|x|^{-(\frac{q+1}{q-1})} & \text{if } \mu \in [\mu^*, \bar{\mu}), \end{cases}$$

for every x satisfying $0 < |x| < \tilde{\rho}$. This completes the theorem. \square

4. HOLDER CONTINUITY AND GREEN FUNCTION ESTIMATES

Lemma 4.1. *Let $R > 0$ be given a small number. Then Green function defined in (1.24) satisfies*

$$(4.1) \quad \sup_{r/2 < |x-y| < r} G(x, y) \leq C \int_r^R \frac{t^2}{w(B_t(x))} \frac{dt}{t}$$

where $w(B_t(x)) = \int_{|x-y| < t} |y|^{-2\nu} dy$ with $N - 2\nu - 2 > 0$ and $r \in (0, \frac{R}{2})$ and $\text{dist}(x, \partial\Omega) > R$, $\text{dist}(y, \partial\Omega) > R$. In fact, we have

$$(4.2) \quad G(x, y) \leq \frac{Cr^2}{w(B_r(x))}.$$

Proof. This is a modification of the theorem by Chanillo-Wheeden [6, pg. 311]. Note that the term in (4.1) contribute when t is close to r . Define $f(y) = |y|^{-2\nu}$. As f is a Muckenhoupt weight it satisfies the doubling property:

$$(4.3) \quad \int_{|x-y| < \frac{r}{2}} f(y) dy \leq \int_{|x-y| < r} f(y) dy \leq C \int_{|x-y| < \frac{r}{2}} f(y) dy.$$

Using this fact we can cut the RHS dia-dically; we are left simply with the term near r and the above expression reduces to

$$\sup_{r/2 < |x-y| < r} G(x, y) \leq C \sum_{k \geq 0}^m \int_{2^k r}^{2^{k+1} r} \frac{t^2}{w(B_t(x))} \frac{dt}{t}.$$

where $2^{m+1}r = R$.

Let

$$(4.4) \quad I = \sum_{k \geq 0} \frac{2^{2k} r^2}{\int_{B_{2^k r}(x)} f(y) dy}.$$

It follows from Lemma B.1 (see Appendix B) that

$$(4.5) \quad \int_{B_{2^k r}(x)} f(y) dy \geq C 2^{k(N-2\nu)} \int_{B_r(x)} f(y) dy$$

where $C > 0$ is independent of x, k and r . Therefore

$$I \leq \frac{Cr^2}{\int_{B_r(x)} f(y) dy} \sum_{k \geq 0} 2^{k(2+2\nu-N)} \leq \frac{Cr^2}{\int_{B_r(x)} f(y) dy} = \frac{Cr^2}{w(B_r(x))}.$$

Hence the lemma follows. \square

Lemma 4.2. *The Green function satisfies the following estimate*

$$(4.6) \quad G(x, y) \leq C \left(\frac{|x|^{2\nu} + |x-y|^{2\nu}}{|x-y|^{N-2}} \right)$$

for any $x \neq y$ and $x, y \in \Omega$ for some $C > 0$ depending on Ω . Moreover, $G(x, \cdot)$ is continuous whenever $x \neq y$.

Proof. Since $\frac{r}{2} < |x-y| < r$, we can write $r = O(|x-y|)$. Then from (4.2), we have

$$(4.7) \quad G(x, y) \leq \frac{C|x-y|^2}{w(B_r(x))}$$

Now we estimate the denominator $D = w(B_r(x))$ in the two cases.

Case 1: $|x-y| \leq \frac{1}{4}|x|$.

In this case we note that

$$D = \int_{|x-y| < r} |y|^{-2\nu} dy = \int_{|y| < r} |x-y|^{-2\nu} dy \geq \omega_N \left(\frac{1}{4}|x| \right)^{-2\nu} r^N \geq C|x|^{-2\nu}|x-y|^N,$$

where ω_N is volume of unit ball in \mathbb{R}^N . Therefore $G(x, y) \leq C \frac{|x|^{2\nu}}{|x-y|^{N-2}}$.

Case 2: $|x-y| > \frac{1}{4}|x|$.

In this case we can write $|y| \leq |x-y| + |x| \leq 5|x-y|$. Therefore

$$D = \int_{|x-y| < r} |y|^{-2\nu} dy \geq \omega_N (5|x-y|)^{-2\nu} r^N = C|x-y|^{-2\nu+N}.$$

Thus $G(x, y) \leq C \frac{|x-y|^{2\nu}}{|x-y|^{N-2}}$. Combining case 1 and case 2, the lemma follows. \square

Lemma 4.3. *Consider the problem*

$$(4.8) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu}\nabla v) = |x|^{-(p+1)\nu}v^p - |x|^{-(q+1)\nu}v^q & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $q < \frac{2+\nu}{\nu}$. Then v is Hölder continuous at the origin with Hölder exponent $\theta = 2 + 2\nu - (q+1)\nu$.

Proof. In order to prove this result we use the information on the Green function. We know that v is bounded at the origin. Also note that near the origin $|x|^{-(q+1)\nu}v^q$ is the dominating term. Then

$$(4.9) \quad v(x) = \int_{\Omega} G(x, z)[|z|^{-(p+1)\nu}v^p(z) - |z|^{-(q+1)\nu}v^q(z)]dz.$$

So we have

$$v(x) - v(y) = \int_{\Omega} [G(x, z) - G(y, z)][|z|^{-(p+1)\nu}v^p(z) - |z|^{-(q+1)\nu}v^q(z)]dz.$$

By the self adjoint-ness of the Green function, for fixed z we consider

$$G(x, z) - G(y, z) = G_z(x) - G_z(y).$$

Consider the ball of radius $|x - y| = \rho$ centered at x . We take $x = 0$. then using the fact that v is bounded and $|y| = \rho$, we obtain

$$|v(y) - v(0)| \leq C(A + B),$$

where

$$A = \int_{|z| \leq \rho} |G_z(0) - G_z(y)| |z|^{-(q+1)\nu} dz$$

and

$$B = \int_{|z| \geq \rho} |G_z(0) - G_z(y)| |z|^{-(q+1)\nu} dz.$$

Case 1 If z lies outside the ball of radius $|y| = \rho$. Then we can think of z as the pole and we are far away. That is, we can consider a ball \mathcal{B} of radius $\frac{|z|}{4}$ centered at 0 such that double the ball does not meet z . Then we can use the Moser–Harnack inequality as in Chanillo–Wheeden [7] to obtain

$$|G_z(0) - G_z(y)| \leq C \left(\frac{|y|}{|z|} \right)^{\delta} \left(\frac{|z|^2}{\int_{B_{\rho}(0)} |\xi|^{-2\nu}} \right) = C \left(\frac{|y|}{|z|} \right)^{\delta} \left(\frac{|z|^2}{|y|^{N-2\nu}} \right),$$

where $\delta > 0$. This is a bound from the Harnack inequality as the Green's function is non-negative. Therefore,

$$B \leq |y|^{\delta-N+2\nu} \int_{|z| \geq \rho} \frac{|z|^2}{|z|^{(q+1)\nu+\delta}} dz = O(|y|^{\theta}).$$

Case 2 In this case, z lies in the ball of radius $|y| = \rho$. Here we use the crude bound on the Green function from Lemma 4.2

$$|G_z(0) - G_z(y)| \leq |G_z(0)| + |G_z(y)| \leq \frac{C}{|z|^{N-2\nu-2}} + |G_z(y)|.$$

Hence we have

$$C \int_{|z| \leq \rho} \frac{|z|^{-(q+1)\nu}}{|z|^{N-2\nu-2}} dz = O(|y|^{2+2\nu-(q+1)\nu}) = O(|y|^{\theta}).$$

Now we estimate $G_z(y)$. For this we divide the domain into two parts. Whenever $|z| \leq \frac{|y|}{2}$, we have $B_{|y|/4}(y) \subset B_{|z-y|}(y)$ and from (4.2)

$$|G_z(y)| \leq C \left(\frac{|z-y|^2}{\int_{B_{|z-y|}(y)} |\xi|^{-2\nu}} \right).$$

As a result we have

$$|G_z(y)| = O(|y|^{2+2\nu-N}).$$

Hence we obtain

$$\int_{|z| \leq \frac{|y|}{2}} |G_z(y)| |z|^{-(q+1)\nu} |v|^q dz \leq c|y|^\theta.$$

We are now left to check what happens in the region $\frac{|y|}{2} \leq |z| \leq |y| = \rho$. So in this region we have

$$(4.10) \quad \int_{\frac{|y|}{2} \leq |z| \leq |y|} |G_z(y)| |z|^{-(q+1)\nu} |v|^q dz \leq C|y|^{-(q+1)\nu} \int_{\frac{|y|}{2} \leq |z| \leq |y|} |G_z(y)| dz.$$

Now suppose that $|z-y| \leq \frac{|y|}{2}$. This implies if $\xi \in B_{|z-y|}(y)$, then we have $|\xi-y| \leq |z-y| \leq \frac{|y|}{2}$. Consequently, $||\xi|-|y|| \leq \frac{|y|}{2}$ which in fact implies that $\frac{|y|}{2} \leq |\xi| \leq \frac{3}{2}|y|$. Therefore

$$\int_{B_{|z-y|}(y)} |\xi|^{-2\nu} d\xi = O(|y|^{-2\nu} |z-y|^N).$$

Using the above expression, we obtain

$$|G_z(y)| \leq C \left(\frac{|z-y|^2}{\int_{B_{|z-y|}(y)} |\xi|^{-2\nu}} \right) = O(|y|^{2\nu} |z-y|^{2-N}).$$

When $|z-y| \geq \frac{|y|}{2}$, then since $|z| \leq |y|$ we have

$$|G_z(y)| = O(|y|^{2+2\nu-N}).$$

Hence from (4.10), we obtain

$$\begin{aligned} \int_{\frac{|y|}{2} \leq |z| \leq |y|} |G_z(y)| |z|^{-(q+1)\nu} |v|^q dz &\leq |y|^{-(q+1)\nu} \int_{\{\frac{|y|}{2} \leq |z| \leq |y|\} \cap \{|z-y| \leq \frac{|y|}{2}\}} G_z(y) dz \\ &+ |y|^{-(q+1)\nu} \int_{\{\frac{|y|}{2} \leq |z| \leq |y|\} \cap \{|z-y| \geq \frac{|y|}{2}\}} G_z(y) dz \\ &\leq C|y|^{2\nu-(q+1)\nu} \int_{\{\frac{|y|}{2} \leq |z| \leq |y|\}} \frac{1}{|z-y|^{N-2}} dz \\ &+ C|y|^{2+2\nu-(q+1)\nu} \\ &= O(|y|^\theta). \end{aligned}$$

□

Lemma 4.4. (*Weighted Pohozaev Identity*) Let Ω be a smooth bounded domain and $v \in C^1(\bar{\Omega} \setminus \{0\}) \cap C^0(\Omega)$ be a positive solution of

$$(4.11) \quad \begin{cases} -\nabla(|x|^{-2\nu} \nabla v) + \varepsilon |x|^{-(q+1)\nu} v^q = |x|^{-(p+1)\nu} v^p & \text{in } \Omega, \\ v \in H^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)}), \end{cases}$$

with $2^* - 1 \leq p < q$, $\nu \in (0, \frac{N-2}{2})$. Then v satisfies

$$\begin{aligned}
& \frac{1}{2} \int_{\partial\Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla v|^2 dS + \left(\frac{N-2-2\nu}{2} \right) \int_{\partial\Omega} |x|^{-2\nu} v \frac{\partial v}{\partial n} dS \\
& - \frac{\varepsilon}{q+1} \int_{\partial\Omega} |x|^{-(q+1)\nu} \langle x, n \rangle v^{q+1} dS + \left(\frac{N}{q+1} - \frac{N-2}{2} \right) \varepsilon \int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx \\
& = \left(\frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx \\
(4.12) \quad & \frac{1}{p+1} \int_{\partial\Omega} |x|^{-(p+1)\nu} \langle x, n \rangle v^{p+1} dS.
\end{aligned}$$

Moreover, if $v = 0$ on $\partial\Omega$, then

$$\begin{aligned}
\frac{1}{2} \int_{\partial\Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla v|^2 dS &= \left(\frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx \\
(4.13) \quad &+ \varepsilon \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx.
\end{aligned}$$

Proof. This follows from Proposition 2.1. Note that, v is a solution of (4.11) implies $u(x) = |x|^{-\nu} v(x)$ is a solution of (2.1). Therefore substituting $u(x) = |x|^{-\nu} v(x)$ in (2.2) we obtain,

$$\begin{aligned}
& \frac{1}{2} \int_{\partial\Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla v|^2 dS + \left(\frac{N-2-2\nu}{2} \right) \int_{\partial\Omega} |x|^{-2\nu} v \frac{\partial v}{\partial n} dS \\
& + \frac{\nu^2 - \nu(N-2) + \mu}{2} \int_{\partial\Omega} |x|^{-2\nu-2} v^2 \langle x, n \rangle dS \\
& = \frac{\varepsilon}{q+1} \int_{\partial\Omega} |x|^{-(q+1)\nu} \langle x, n \rangle v^{q+1} dS - \varepsilon \left(\frac{N}{q+1} - \frac{N-2}{2} \right) \int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx \\
& + \left(\frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx - \frac{1}{p+1} \int_{\partial\Omega} |x|^{-(p+1)\nu} \langle x, n \rangle v^{p+1} dS.
\end{aligned}$$

Since $\nu^2 - \nu(N-2) + \mu = 0$, the above expression reduces to (4.12). Consequently $v = 0$ on $\partial\Omega$ implies (4.13). \square

Lemma 4.5. Let $\nu \in (0, \frac{N-2}{2})$. Then the Green function $G(x, 0)$ satisfies

$$(4.14) \quad \int_{\partial\Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla G(x, 0)|^2 dS = (N-2-2\nu) |R(0)|.$$

Proof. We apply Pohozaev identity (4.12) to (1.24) on $\Omega \setminus B_r(0)$, for r sufficiently small. Then we have

$$\begin{aligned}
\int_{\partial\Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla G(x, 0)|^2 dS &= \int_{\partial B_r} |x|^{-2\nu} \langle x, n \rangle |\nabla G(x, 0)|^2 dS \\
&+ (N-2-2\nu) \int_{\partial B_r} |x|^{-2\nu} G(x, 0) \frac{\partial G(x, 0)}{\partial n} \\
&= r \int_{\partial B_r} |x|^{-2\nu} |\nabla G(x, 0)|^2 dS \\
(4.15) \quad &+ (N-2-2\nu) \int_{\partial B_r} |x|^{-2\nu} G(x, 0) \frac{\partial G(x, 0)}{\partial n}
\end{aligned}$$

Moreover, from (1.23) and (1.26) we have,

$$(4.16) \quad G(x, 0) = H(x, 0) + \frac{1}{(N - 2\nu - 2)\omega_N |x - y|^{N-2\nu-2}}$$

and hence

$$(4.17) \quad \nabla G(x, 0) = -\frac{1}{\omega_N} |x|^{(2\nu-N)} x + \nabla H(x, 0).$$

Substituting $G(x, 0)$ and $\nabla G(x, 0)$ in (4.15), we take the limit $r \rightarrow 0$. After simplifying the terms, we obtain

$$(4.18) \quad \begin{aligned} \lim_{r \rightarrow 0} \text{RHS of (4.15)} &= \lim_{r \rightarrow 0} r^{-2\nu-1} \int_{\partial B_r} |x \cdot \nabla H(x, 0)|^2 dS \\ &- \frac{r^{1-N}}{\omega_N} \int_{\partial B_r} \langle x \cdot \nabla H(x, 0) \rangle dS \\ &- \frac{(N - 2 - 2\nu)}{\omega_N} r^{-N+1} \int_{\partial B_r} H(x, 0) dS \\ &+ (N - 2 - 2\nu) r^{-2\nu-1} \int_{\partial B_r} H(x, 0) \langle x \cdot \nabla H(x, 0) \rangle dS \end{aligned}$$

Note that, as $H(x, 0)$ is Holder continuous at origin [6], it follows $|x \cdot \nabla H(x, 0)| \rightarrow 0$ on ∂B_r as $r \rightarrow 0$. Therefore a straight forward computation yields

$$\text{RHS of (4.15)} = -\frac{(N - 2 - 2\nu)}{\omega_N r^{N-1}} \int_{\partial B_r} H(x, 0) dS.$$

Using the mean value theorem,

$$(4.19) \quad R(0) = H(0, 0) = \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r} H(x, 0) dS.$$

Hence the lemma follows. \square

5. SYMMETRY AND DECAY PROPERTIES OF ENTIRE PROBLEM

In this section using moving plane method, we give the proof of Theorem 1.9.

Proof of Theorem 1.9. It is enough to show that u is symmetric with respect to each coordinate axis. For $\alpha > 0$, we define

$$\Omega_\alpha = \{x \in \mathbb{R}^N : x_1 > \alpha\},$$

and for $x \in \Omega_\alpha$, let x_α denote the it's reflection to the hyperplane $x_1 = \alpha$, that is $x_\alpha = (2\alpha - x_1, x_2, \dots, x_n)$. Set

$$u_\alpha(x) := u(x_\alpha), \quad x \in \Omega_\alpha \quad \text{and} \quad w_\alpha = u_\alpha - u.$$

We note that w_α is smooth away from the point $(2\alpha, 0, \dots, 0)$ and $w_\alpha = 0$ on $\partial\Omega_\alpha$. It is easy to check that $w_\alpha \in D^{1,2}(\Omega_\alpha)$.

Claim 1: $w_\alpha \geq 0$ in Ω_α , if $\alpha > 0$ is large enough.

To see the claim, we note that $|x_\alpha| < |x|$ if $\alpha > 0$. By a straight forward computation it follows that w_α satisfies the following equation

$$(5.1) \quad -\Delta w_\alpha - \mu \frac{w_\alpha}{|x|^2} \geq A_1(x) w_\alpha - A_2(x) w_\alpha \quad \text{in } \Omega_\alpha,$$

where

$$0 \leq A_1(x) := \lambda \frac{u_\alpha^p - u^p}{u_\alpha - u} \leq \lambda p [\max\{u_\alpha(x), u(x)\}]^{p-1}$$

and

$$0 \leq A_2(x) := \frac{u_\alpha^q - u^q}{u_\alpha - u} \leq q [\max\{u_\alpha(x), u(x)\}]^{q-1}.$$

Multiplying (5.1) by w_α^- and integrating by parts over Ω_α , we obtain

$$\begin{aligned} \int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 dx - \mu \frac{|w_\alpha^-|^2}{|x|^2} dx &\leq \int_{\Omega_\alpha} (A_1(x) - A_2(x)) |w_\alpha^-|^2 dx \\ &\leq \int_{\Omega_\alpha} A_1(x) |w_\alpha^-|^2 dx \\ (5.2) \quad &\leq \left(\int_{\Omega_\alpha} |w_\alpha^-|^{2^*} dx \right)^{\frac{N-2}{N}} \left(\int_{\Omega_\alpha \cap \{w_\alpha < 0\}} A_1^{\frac{N}{2}} dx \right)^{\frac{2}{N}}. \end{aligned}$$

As $\mu < (\frac{N-2}{2})^2$, it is not difficult to check that $\left(\int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 - \mu \frac{|w_\alpha^-|^2}{|x|^2} \right)^{\frac{1}{2}}$ is an equivalent norm to $D^{1,2}(\mathbb{R}^N)$. Therefore there exists a positive constant C_1 such that $C_1 \int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 \leq \int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 - \mu \frac{|w_\alpha^-|^2}{|x|^2}$. Applying this estimate along with Sobolev inequality, we have from (5.2):

$$(5.3) \quad C_1 \mathcal{S} \left(\int_{\Omega_\alpha} |w_\alpha^-|^{2^*} dx \right)^{\frac{2}{2^*}} \leq \left(\int_{\Omega_\alpha} |w_\alpha^-|^{2^*} dx \right)^{\frac{N-2}{N}} \left(\int_{\Omega_\alpha \cap \{w_\alpha < 0\}} A_1^{\frac{N}{2}} dx \right)^{\frac{2}{N}},$$

where \mathcal{S} is the Sobolev constant. On the other hand, $u_\alpha < u$ on $\{w_\alpha < 0\}$ implies

$$\int_{\Omega_\alpha \cap \{w_\alpha < 0\}} A_1^{\frac{N}{2}} dx \leq C \int_{\Omega_\alpha \cap \{w_\alpha < 0\}} u^{(p-1)\frac{N}{2}}.$$

We know that $u \in L^{2^*}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$. As by the given assumption

$$q > (p-1)\frac{N}{2} - 1 \quad \text{and} \quad p \geq 2^* - 1,$$

using interpolation theory we can show that $u \in L^{(p-1)\frac{N}{2}}$. Consequently

$$\int_{\Omega_\alpha \cap \{w_\alpha < 0\}} u^{(p-1)\frac{N}{2}} \rightarrow 0 \quad \text{if } \alpha \text{ is large enough.}$$

Hence from (5.3) we conclude $w_\alpha^- = 0$ in Ω_α if α is large enough. This proves the claim.

Let

$$\alpha_0 = \inf\{\alpha > 0 : u_{\alpha'} \geq u \text{ in } \Omega_{\alpha'} \forall \alpha' > \alpha\}.$$

Claim 2: $\alpha_0 = 0$.

We will prove this claim by method of contradiction. Let $\alpha_0 > 0$. Define $w_{\alpha_0} = u_{\alpha_0} - u$. Then $w_{\alpha_0} \geq 0$ in Ω_{α_0} and $-\Delta w_{\alpha_0} + A_2(x)w_{\alpha_0} = \mu \frac{w_{\alpha_0}}{|x|^2} + A_1(x)w_{\alpha_0} \geq 0$ in Ω_{α_0} and away from the point $(2\alpha_0, 0, \dots, 0)$. As $A_2 \geq 0$, by maximum principle we have $w_{\alpha_0} > 0$ in this region.

Let $\epsilon > 0$. We choose $R > 0$ and $\delta_0 > 0$ such that

$$(5.4) \quad \int_{|x| > R} u^{(p-1)\frac{N}{2}} dx < \frac{\epsilon}{2}$$

and

$$(5.5) \quad \int_{\alpha_0 - \delta_0 < x_1 < \alpha_0 + \delta_0} u^{(p-1)\frac{N}{2}} dx + \int_{2\alpha_0 - \delta_0 < x_1 < 2\alpha_0 + \delta_0} u^{(p-1)\frac{N}{2}} dx < \frac{\epsilon}{2}.$$

Define

$$K := \{x \in \Omega : \alpha_0 + \delta_0 \leq x_1 \leq 2\alpha_0 - \delta_0 \text{ or } x_1 \geq 2\alpha_0 + \delta_0\} \cap \{|x| \leq R\}.$$

Then K is a compact set and $w_{\alpha_0} > 0$ in K . Choose $\delta_1 \in (0, \delta_0)$ such that $w_{\alpha_0 - \delta} > 0$ in $K \quad \forall \delta \in (0, \delta_1)$. Define $\alpha_1 := \alpha_0 - \delta$. Next we will show that $u_{\alpha_1} \geq u$ in Ω_{α_1} and this will contradict the definition of α_0 . Towards this goal, we define $w_{\alpha_1} := u_{\alpha_1} - u$. We proceed as in the case of (5.3) to get

$$C_1 \mathcal{S} \left(\int_{\Omega_{\alpha_1}} |w_{\alpha_1}^-|^{2^*} dx \right)^{\frac{2}{2^*}} \leq \left(\int_{\Omega_{\alpha_1}} |w_{\alpha_1}^-|^{2^*} dx \right)^{\frac{2}{2^*}} \left(\int_{\Omega_{\alpha_1} \cap \{w_{\alpha_1} < 0\}} A_1^{\frac{N}{2}} dx \right)^{\frac{2}{N}}.$$

By the choice of α_1 , we have $w_{\alpha_1} > 0$ in K and thus by (5.4) and (5.5), we conclude that

$$\int_{\Omega_{\alpha_1} \cap \{w_{\alpha_1} < 0\}} A_1^{\frac{N}{2}} dx \leq \lambda p \int_{\Omega_{\alpha_1} \cap \{w_{\alpha_1} < 0\}} u^{(p-1)\frac{N}{2}} dx < \epsilon.$$

As $\epsilon > 0$ is arbitrarily chosen, we can conclude that $w_{\alpha_1}^- = 0$, which contradicts the definition of α_0 . Hence the claim follows.

Consequently we have

$$u(-x_1, x_2, \dots, x_n) \geq u(x_1, x_2, \dots, x_n) \quad \forall x_1 > 0.$$

Now repeating the same arguments for $\tilde{u}(x) = u(-x_1, x_2, \dots, x_n)$, we can prove that

$$u(-x_1, x_2, \dots, x_n) \leq u(x_1, x_2, \dots, x_n) \quad \forall x_1 > 0.$$

Hence

$$u(-x_1, x_2, \dots, x_n) = u(x_1, x_2, \dots, x_n) \quad \forall x_1 > 0.$$

As a result symmetry follows since we can do the moving plane argument in any direction instead of x_1 direction. \square

Remark 5.1. *Doing some simple modifications to the proof of Theorem 1.9, it can be shown that u is a radially symmetric solution of (1.1), when $\Omega = B_R(0)$, for any $R > 0$. Therefore v is a radially symmetric solution of (1.2), when $\Omega = B_R(0)$, for any $R > 0$.*

From Theorem 1.2, we know \mathcal{K} is achieved by a radial function $v \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu})$. Furthermore, there exists a constant $\lambda > 0$ such that v satisfies the following problem:

$$(5.6) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla v) = \lambda |x|^{-(p+1)\nu} v^p - |x|^{-(q+1)\nu} v^q & \text{in } \mathbb{R}^N, \\ v > 0 & \text{in } \mathbb{R}^N, \\ v \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu}). \end{cases}$$

Lemma 5.1. *Define $\alpha = N - 2 - 2\nu$. Let $2^* - 1 < p$ and $q > (p-1)\frac{N}{2} - 1$, $0 < \nu < \frac{N-2}{2}$. Suppose v is a solution of (5.6). Then $v(x) = v(|x|) = v(r)$, where $|x| = r$. Moreover,*

$$\int_0^\infty (v(r))^{2^*} r^{N-1-2^*\nu} dr < +\infty.$$

Furthermore, if $v(r) \leq Cr^{-\alpha}$ for $r \gg 1$ then $v \sim r^{-\alpha}$ as $r \rightarrow \infty$.

Proof. If v is any solution of (5.6), then $u = |x|^{-\nu}v$ is a solution of (1.35). By Theorem 1.9, u is radially symmetric. Therefore $v(x) = v(|x|) = v(r)$ and v satisfies the following ode

$$(5.7) \quad \begin{cases} v_{rr} + \frac{(N-2\nu-1)}{r}v_r + \lambda r^{-(p-1)\nu}v^p - r^{-(q-1)\nu}v^q = 0 & \text{in } (0, \infty), \\ v(r) > 0 & \text{in } (0, \infty), \\ \int_0^\infty (v'(r))^2 r^{N-1-2\nu} dr < +\infty. \end{cases}$$

Applying Sobolev embedding theorem it follows $\int_0^\infty (v(r))^{2^*} r^{N-1-2^*\nu} dr < +\infty$.

To prove the last assertion, we first show that $v(r) \geq Cr^{-\alpha}$ for some $C > 0$ and $r \gg 1$.

To see this, let $w = |x|^{-\alpha}$. Then $\operatorname{div}(|x|^{-2\nu}\nabla w) = 0$ in $\mathbb{R}^N \setminus B_R(0)$. Hence by standard method using comparison principle it follows that

$$v \geq Cw = C|x|^{-\alpha} \quad \text{in } \mathbb{R}^N \setminus B_R(0).$$

Hence there exist $C_1, C_2 > 0$ such that

$$(5.8) \quad C_1 \leq v(r)r^\alpha \leq C_2 \text{ for } r \gg 1.$$

But from (5.7) we have

$$-(r^{N-1-2\nu}v_r)_r = v^p r^{N-1-(p+1)\nu}(1 + o(1)) \quad \text{for } r \gg 1.$$

Since v satisfies (5.8) and $p > 2^* - 1$, the RHS of the above expression is integrable in (s, ∞) and positive. This implies that

$$\lim_{r \rightarrow \infty} v_r r^{(N-1-2\nu)} = -c.$$

for some $c > 0$. This in fact implies that $v_r \sim -r^{-(N-1-2\nu)}$. Integrating this expression from (s, ∞) we obtain,

$$\lim_{r \rightarrow \infty} r^{N-2-2\nu}v = a \in (0, +\infty).$$

□

6. PROOF OF THEOREM 1.10

6.1. Auxiliary results. Define

$$(6.1) \quad \hat{F}(w) = \frac{1}{2} \int |x|^{-2\nu} |\nabla w|^2 dx + \frac{1}{q+1} \int |x|^{-(q+1)\nu} w^{q+1} dx,$$

where $\nu \in (0, \frac{N-2}{2})$, $q > p \geq 2^* - 1$. For $\rho > 0$, set

$$N_\rho = \left\{ w \in H_0^1(\rho\Omega, |x|^{-2\nu}) \cap L^{q+1}(\rho\Omega, |x|^{-(q+1)\nu}) : \int_{\rho\Omega} |x|^{-(p+1)\nu} w^{p+1} dx = 1 \right\}.$$

Define

$$S_\rho := \inf_{w \in N_\rho} \hat{F}(w).$$

Theorem 6.1. *Let $p = 2^* - 1$. Then $S_\rho \rightarrow \frac{\mathcal{S}}{2}$ as $\rho \rightarrow \infty$, where \mathcal{S} is as defined in (1.19).*

Proof. Step 1: $\lim_{\rho \rightarrow \infty} S_\rho \leq \frac{S}{2}$

To see this, let $U(x)$ be as in (1.20). We know from [5] that, U is an extremal of \mathcal{S} , with $\int_{\mathbb{R}^N} |x|^{2^* \nu} U^{2^*}(x) dx = 1$ and U is a ground state solution of (1.22). It is easy to check that $\mu^{-\frac{\alpha}{2}} U(\frac{x}{\mu})$ is also a solution of (1.22), for any $\mu > 0$, where $\alpha = N - 2 - 2\nu$.

Set $\rho := \mu^2$. Define

$$U_\rho(x) := \mu^{-\frac{\alpha}{2}} U\left(\frac{x}{\mu}\right) \quad \text{and} \quad \phi_\rho(x) = \phi\left(\frac{x}{\rho}\right),$$

where $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\text{supp } \phi \in \Omega$, $\phi = 1$ in $\frac{\Omega}{2}$, $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq \frac{2}{d}$ and $d = \text{diam}(\Omega)$. We set

$$v_\rho(x) := U_\rho(x) \phi_\rho(x) \quad \text{and} \quad \hat{v}_\rho := \frac{v_\rho}{\| |x|^{-\nu} v_\rho \|_{L^{2^*}(\rho\Omega)}}.$$

Then $\hat{v}_\rho \in N_\rho$.

$$\begin{aligned} \lim_{\rho \rightarrow \infty} \int_{\rho\Omega} |x|^{-2^* \nu} v_\rho^{2^*} dx &= \lim_{\rho \rightarrow \infty} \mu^{-\frac{\alpha}{2} 2^*} \int_{\mathbb{R}^N} |x|^{-2^* \nu} U^{2^*}\left(\frac{x}{\mu}\right) \phi^{2^*}\left(\frac{x}{\rho}\right) dx \\ &= \lim_{\rho \rightarrow \infty} \rho^{-\frac{\alpha}{2} 2^* + N - 2^* \nu} \int_{\mathbb{R}^N} |x|^{-2^* \nu} U^{2^*}(x) \phi^{2^*}\left(\frac{x}{\sqrt{\rho}}\right) dx \\ &= \lim_{\rho \rightarrow \infty} \int_{\mathbb{R}^N} |x|^{-2^* \nu} U^{2^*}(x) \phi^{2^*}\left(\frac{x}{\sqrt{\rho}}\right) dx \\ (6.2) \quad &= \int_{\mathbb{R}^N} |x|^{-2^* \nu} U^{2^*}(x) dx = 1. \end{aligned}$$

Similarly we see that

$$\int_{\rho\Omega} |x|^{-(q+1)\nu} \hat{v}_\rho^{q+1} dx = \frac{\rho^{-\frac{\alpha}{2}(q+1) + N - (q+1)\nu}}{\| |x|^{-\nu} v_\rho \|_{L^{2^*}(\rho\Omega)}^{q+1}} \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} U^{q+1}(x) \phi^{q+1}\left(\frac{x}{\sqrt{\rho}}\right) dx.$$

As before

$$\lim_{\rho \rightarrow \infty} \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} U^{q+1}(x) \phi^{q+1}\left(\frac{x}{\sqrt{\rho}}\right) dx = \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} U^{q+1}(x) dx.$$

Moreover $q > 2^* - 1$ implies $-\frac{\alpha}{2}(q+1) + N - (q+1)\nu < 0$. Hence by (6.2), we have

$$(6.3) \quad \lim_{\rho \rightarrow \infty} \int_{\rho\Omega} |x|^{-(q+1)\nu} \hat{v}_\rho^{q+1} dx = 0.$$

(6.4)

$$\int_{\rho\Omega} |x|^{-2\nu} |\nabla \hat{v}_\rho|^2 dx = \frac{\int_{\rho\Omega} |x|^{-2\nu} |\nabla v_\rho|^2 dx}{\| |x|^{-\nu} v_\rho \|_{L^{2^*}(\rho\Omega)}^2}; \quad \text{and} \quad \int_{\rho\Omega} |x|^{-2\nu} |\nabla v_\rho|^2 dx = I_\rho^1 + I_\rho^2 + I_\rho^3,$$

where

$$\begin{aligned} I_\rho^1 &= \mu^{-(\alpha+2)} \int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U\left(\frac{x}{\mu}\right)|^2 \phi^2\left(\frac{x}{\rho}\right) dx; \\ I_\rho^2 &= \mu^{-(\alpha+4)} \int_{\rho\Omega \setminus \rho\frac{\Omega}{2}} |x|^{-2\nu} U^2\left(\frac{x}{\mu}\right) |\nabla \phi\left(\frac{x}{\rho}\right)|^2 dx; \\ I_\rho^3 &= 2\mu^{-(\alpha+3)} \int_{\rho\Omega \setminus \rho\frac{\Omega}{2}} |x|^{-2\nu} U\left(\frac{x}{\mu}\right) \phi\left(\frac{x}{\rho}\right) \nabla U\left(\frac{x}{\mu}\right) \nabla \phi\left(\frac{x}{\rho}\right) dx. \end{aligned}$$

By straight forward computation we see that

$$(6.5) \quad \lim_{\rho \rightarrow \infty} I_\rho^1 = \lim_{\rho \rightarrow \infty} \int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U(x)|^2 \phi^2\left(\frac{x}{\sqrt{\rho}}\right) dx = \int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U(x)|^2 dx = \mathcal{S}.$$

$$(6.6) \quad \begin{aligned} \lim_{\rho \rightarrow \infty} I_\rho^2 &\leq \lim_{\rho \rightarrow \infty} \frac{4}{d^2} \mu^{-(\alpha+4)} \int_{\rho\Omega \setminus \rho\frac{\Omega}{2}} |x|^{-2\nu} U^2\left(\frac{x}{\mu}\right) dx \\ &= \lim_{\rho \rightarrow \infty} \frac{4}{d^2} \mu^{-(\alpha+4)-2\nu+N} \int_{\sqrt{\rho}\Omega \setminus \sqrt{\rho}\frac{\Omega}{2}} |x|^{-2\nu} U^2(x) dx \\ &\leq \lim_{\rho \rightarrow \infty} \frac{4}{d^2} \mu^{-(\alpha+4)-2\nu+N} \left(\int_{\mathbb{R}^N} |x|^{-2*\nu} U^{2^*}(x) dx \right)^{\frac{2}{2^*}} |\sqrt{\rho}(\Omega \setminus \frac{\Omega}{2})|^{1-\frac{2}{2^*}} \\ &\leq \lim_{\mu \rightarrow \infty} C \mu^{-(\alpha+4)-2\nu+N+(1-\frac{2}{2^*})} = \lim_{\mu \rightarrow \infty} C \mu^{-1-\frac{2}{2^*}} = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{\rho \rightarrow \infty} I_\rho^3 &\leq \lim_{\rho \rightarrow \infty} \mu^{-(\alpha+3)} \left(\int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U\left(\frac{x}{\mu}\right)|^2 \phi^2\left(\frac{x}{\rho}\right) dx \right)^{\frac{1}{2}} \times \\ &\quad \left(\int_{\rho\Omega \setminus \rho\frac{\Omega}{2}} |x|^{-2\nu} |\nabla \phi\left(\frac{x}{\rho}\right)|^2 \phi^2\left(\frac{x}{\mu}\right) dx \right)^{\frac{1}{2}}. \end{aligned}$$

Consequently,

$$(6.7) \quad \lim_{\rho \rightarrow \infty} I_\rho^3 \leq \lim_{\mu \rightarrow \infty} C \mu^{-\frac{2}{2^*}} = 0.$$

Combining (6.3), (6.5), (6.6), (6.7) and (6.2) we obtain

$$S_\rho \leq F(v_\rho) \quad \text{and} \quad F(v_\rho) \rightarrow \frac{\mathcal{S}}{2} \quad \text{as} \quad \rho \rightarrow \infty.$$

Hence

$$(6.8) \quad \lim_{\rho \rightarrow \infty} S_\rho \leq \frac{\mathcal{S}}{2}.$$

Step 2: $\frac{\mathcal{S}}{2} \leq \lim_{\rho \rightarrow \infty} S_\rho$.

This is standard to prove. Therefore we just give here a sketch of the proof. Let $\varepsilon > 0$. Then there exists $u_{\rho,\varepsilon} \in N_\rho$ such that

$$(6.9) \quad \hat{F}(u_{\rho,\varepsilon}) < S_\rho + \varepsilon.$$

Extend $u_{\rho,\varepsilon}$ by 0 outside $\rho\Omega$ and we denote it by $u_{\rho,\varepsilon}$ too. Let $\eta(x) = C \exp(\frac{1}{|x|^2-1})$ if $|x| < 1$ and 0 otherwise. Set $\eta_\delta(x) = \delta^{-N} \eta(\frac{x}{\delta})$.

Define $u_{\rho,\varepsilon}^\delta := u_{\rho,\varepsilon} * \eta_\delta$ and $v_{\rho,\varepsilon}^\delta = \frac{u_{\rho,\varepsilon}^\delta}{|u_{\rho,\varepsilon}^\delta|_{L^{2^*}(\mathbb{R}^N)}}$. Thus $v_{\rho,\varepsilon}^\delta \in C_0^\infty(\mathbb{R}^N) \cap N$, where

$$N := \left\{ w \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu} dx) : w \in L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu} dx), \int_{\mathbb{R}^N} |x|^{-2*\nu} w^{2^*} dx = 1 \right\}.$$

Moreover,

$$v_{\rho,\varepsilon}^\delta \rightarrow u_{\rho,\varepsilon} \quad \text{in} \quad D^{1,2}(\mathbb{R}^N, |x|^{-2\nu} dx) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu} dx) \quad \text{as} \quad \delta \rightarrow 0.$$

Hence

$$\frac{\mathcal{S}}{2} \leq \hat{F}(v_{\rho,\varepsilon}^\delta) \rightarrow \hat{F}(u_{\rho,\varepsilon}) \quad \text{as} \quad \delta \rightarrow 0.$$

Combining this with (6.9), we conclude $\frac{S}{2} < S_\rho + \varepsilon$. As $\varepsilon > 0$ is arbitrary, this proves Step 2.

Combining Step 1 and Step 2, theorem follows. \square

Theorem 6.2. *Let $p > 2^* - 1$. Then $S_\rho \rightarrow \mathcal{K}$ as $\rho \rightarrow \infty$, where \mathcal{K} is as defined in (1.28).*

Proof. Let $w \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu} dx) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu} dx)$ be a minimizer of \mathcal{K} (which exists by Theorem 1.2) with $\int_{\mathbb{R}^N} |x|^{-(p+1)\nu} w^{p+1} dx = 1$. Define ϕ_ρ as in Step 1 of the proof of Theorem 6.1. Set $w_\rho = w\phi_\rho$ and $\hat{w}_\rho = \frac{w_\rho}{\| |x|^{-\nu} w_\rho \|_{L^{p+1}(\mathbb{R}^N)}}$. Then $\hat{w}_\rho \in N_\rho$ and consequently $S_\rho \leq F(\hat{w}_\rho)$. Proceeding the same way as in Step 1 of Theorem 6.1, we obtain $F(\hat{w}_\rho) \rightarrow \mathcal{K}$ as $\rho \rightarrow \infty$. Hence $\lim_{\rho \rightarrow \infty} S_\rho \leq \mathcal{K}$. To get the other sided inequality we use the same idea as in Step 2 of Theorem 6.2. This completes the proof. \square

6.2. Asymptotic Behavior. For $v \in H_0^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu})$, we recall the definition of the functional $F(\cdot, \Omega)$ from (1.28) for $p > 2^* - 1$ and $S(\cdot, \Omega)$ from (1.31) for $p = 2^* - 1$:

$$F(v, \Omega) = \frac{1}{2} \frac{\int_{\Omega} |x|^{-2\nu} |\nabla v|^2 dx}{\int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx} + \frac{1}{q+1} \frac{\int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx}{\left(\int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx \right)^l},$$

where

$$l = \frac{2(q+1) - N(p-1)}{2(p+1) - N(p-1)}, \quad q > p > 2^* - 1.$$

$$S(v) = \frac{\int_{\Omega} |x|^{-2\nu} |\nabla v|^2 dx}{\left(\int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx \right)^{\frac{2}{p+1}}}, \quad p = 2^* - 1.$$

Using the transform

$$(6.10) \quad v(x) = \varepsilon^{-\frac{2+2\nu-(p+1)\nu}{2(q-p)}} w(\varepsilon^{-\frac{p-1}{2(q-p)}} x),$$

Eq. (1.2) reduces to

$$(6.11) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla w) = |x|^{-(p+1)\nu} w^p - |x|^{-(q+1)\nu} w^q & \text{in } \Omega_\varepsilon, \\ w > 0 & \text{in } \Omega_\varepsilon, \\ w(x) = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \frac{\Omega}{\varepsilon^{\frac{p-1}{2(q-p)}}}$. Clearly $\Omega_\varepsilon \mapsto \mathbb{R}^N$ as $\varepsilon \rightarrow 0$.

Proposition 6.1. *Let $2^* - 1 \leq p < q$ and $\nu \in (0, \frac{N-2}{2})$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the problem*

$$(6.12) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla v) = \lambda_\varepsilon |x|^{-(p+1)\nu} v^p - \varepsilon |x|^{-(q+1)\nu} v^q & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a solution v_ε , with the property that

$$A < \lambda_\varepsilon < B,$$

for some constants $A, B > 0$, independent of n . In addition

$$(i) \text{ if } p > 2^* - 1, \text{ then } F(v_\varepsilon) \rightarrow \mathcal{K} \text{ and } \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0;$$

$$(ii) \text{ if } p = 2^* - 1, \text{ then } S(v_\varepsilon) \rightarrow \mathcal{S} \text{ as } \varepsilon \rightarrow 0 \text{ and } \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} dx = 1,$$

where \mathcal{K} and \mathcal{S} are defined as in (1.30) and (1.19) respectively.

Proof. Let $\Omega_\varepsilon = \frac{\Omega}{\varepsilon^{\frac{p-1}{2(q-p)}}}$. We are going to work on the manifold

$$N_\varepsilon = \left\{ w \in H_0^1(\Omega_\varepsilon, |x|^{-2\nu}) \cap L^{q+1}(\Omega_\varepsilon, |x|^{-(q+1)\nu}) : \int_{\Omega_\varepsilon} |x|^{-(p+1)\nu} w^{p+1} dx = 1 \right\}.$$

Then F on N_ε reduces to \hat{F} (defined as in Subsection 6.1)

$$F(w) = \frac{1}{2} \int_{\Omega_\varepsilon} |x|^{-2\nu} |\nabla w|^2 dx + \frac{1}{q+1} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} w^{q+1} dx = \hat{F}(w).$$

For every $p \geq 2^* - 1$, let

$$(6.13) \quad S_\varepsilon = \inf_{N_\varepsilon} \hat{F}(w) = \inf_{N_\varepsilon} F(w).$$

Let $\{w_{n,\varepsilon}\}$ be a minimising sequence in N_ε such that

$$\hat{F}(w_{n,\varepsilon}) \rightarrow S_\varepsilon \text{ with } \int_{\Omega_\varepsilon} |x|^{-(p+1)\nu} w_{n,\varepsilon}^{p+1} dx = 1.$$

Thus $\{w_{n,\varepsilon}\}$ is bounded in $H_0^1(\Omega_\varepsilon, |x|^{-2\nu}) \cap L^{q+1}(\Omega_\varepsilon, |x|^{-(q+1)\nu})$. Hence $w_{n,\varepsilon} \rightharpoonup w_\varepsilon$ in $H_0^1(\Omega_\varepsilon, |x|^{-2\nu})$ and $w_{n,\varepsilon} \rightarrow w_\varepsilon$ in $L^2(\Omega_\varepsilon, |x|^{-2\nu})$. As a result, $w_{n,\varepsilon} \rightarrow w_\varepsilon$ pointwise almost everywhere. By the interpolation inequality, we have $w_{n,\varepsilon} \rightarrow w$ on $L^{p+1}(\Omega_\varepsilon, |x|^{-(p+1)\nu})$. Consequently $\int_{\Omega_\varepsilon} |x|^{-2\nu} w_\varepsilon^{p+1} dx = 1$.

Now we show that $S_\varepsilon = \hat{F}(w_\varepsilon)$. Clearly $S_\varepsilon \leq \hat{F}(w_\varepsilon)$. Furthermore, applying Fatou's Lemma and the fact that $w \mapsto \|w\|_{H_0^1(\Omega_\varepsilon, |x|^{-2\nu})}^2$ is weakly lower semicontinuous, we have

$$\begin{aligned} S_\varepsilon &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \int_{\Omega_\varepsilon} |x|^{-2\nu} |\nabla w_{n,\varepsilon}|^2 dx + \frac{1}{q+1} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} w_{n,\varepsilon}^{q+1} dx \right] \\ &\geq \left[\frac{1}{2} \int_{\Omega_\varepsilon} |x|^{-2\nu} |\nabla w_\varepsilon|^2 dx + \frac{1}{q+1} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} w_\varepsilon^{q+1} dx \right] \\ &\geq \hat{F}(w_\varepsilon). \end{aligned}$$

Hence S_ε is achieved by w_ε .

Using the Lagrange multiplier rule, we obtain w_ε satisfies

$$(6.14) \quad -\operatorname{div}(|x|^{-2\nu} \nabla w_\varepsilon) = \lambda_\varepsilon |x|^{-(p+1)\nu} w_\varepsilon^p - |x|^{-(q+1)\nu} w_\varepsilon^q \text{ in } \Omega_\varepsilon,$$

where $\lambda_\varepsilon = \lambda(\varepsilon)$. Moreover,

$$\int_{\Omega_\varepsilon} |x|^{-2\nu} |\nabla w_\varepsilon|^2 dx = \lambda_\varepsilon \int_{\Omega_\varepsilon} |x|^{-(p+1)\nu} w_\varepsilon^{p+1} dx - \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} w_\varepsilon^{q+1} dx,$$

which implies that

$$\lambda_\varepsilon = \int_{\Omega_\varepsilon} |x|^{-2\nu} |\nabla w_\varepsilon|^2 dx + \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} w_\varepsilon^{q+1} dx.$$

This fact along with $\hat{F}(w_\varepsilon) = S_\varepsilon$ implies

$$2S_\varepsilon < \lambda_\varepsilon < (q+1)S_\varepsilon.$$

In Theorem 6.1 and 6.2, if we take $\rho = \varepsilon^{-\frac{p-1}{2(q-p)}}$, then N_ρ and S_ρ of those theorems reduces to N_ε and S_ε defined as above. Therefore taking the limit $\varepsilon \rightarrow 0$, it follows from Theorem 6.1 and 6.2 that

$$(6.15) \quad S_\varepsilon \rightarrow \mathcal{K} \quad \text{if } p > 2^* - 1 \quad \text{and} \quad S_\varepsilon \rightarrow \frac{\mathcal{S}}{2} \quad \text{if } p = 2^* - 1.$$

Hence there exist constants $\varepsilon_0 > 0$ and $A, B > 0$ such that

$$A < \lambda_\varepsilon < B \quad \forall \quad \varepsilon \in (0, \varepsilon_0).$$

Using the transformation (6.10), we obtain from (6.14) that v_ε is a solution of (6.12).

Moreover $\int_{\Omega_\varepsilon} |x|^{-(p+1)\nu} w_\varepsilon^{p+1} dx = 1$ implies $\int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} dx = \varepsilon^{\frac{p(N-2)-(N+2)}{2(q-p)}}$.

Hence

$$\int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} dx = 1 \quad \text{when } p = 2^* - 1$$

and

$$\int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{when } p > 2^* - 1.$$

By a straight forward computation we see that

$$F(w_\varepsilon) = \hat{F}(w_\varepsilon) = F(v_\varepsilon), \quad \text{when } p > 2^* - 1$$

where F and \hat{F} are as in (1.28) and (6.1) respectively. This along with (6.15) and the fact that $F(w_\varepsilon) = S_\varepsilon$ implies

$$F(v_\varepsilon) \rightarrow \mathcal{K} \quad \text{if } p > 2^* - 1$$

Moreover when $p = 2^* - 1$,

$$\mathcal{S} \leq S(v_\varepsilon) \leq 2\hat{F}(v_\varepsilon, \Omega) = 2\hat{F}(w_\varepsilon, \Omega_\varepsilon) = 2S_\varepsilon \rightarrow \mathcal{S}.$$

Hence

$$S(v_\varepsilon) \rightarrow \mathcal{S} \quad \text{if } p = 2^* - 1.$$

This completes the proof. \square

Proof of Theorem 1.10. Let v_ε and λ_ε be as in Proposition 6.1. Setting $u_\varepsilon = \lambda_\varepsilon^{\frac{1}{p-1}} v_\varepsilon$, we find u_ε satisfies

$$-div(|x|^{-2\nu} \nabla u_\varepsilon) = |x|^{-(p+1)\nu} u_\varepsilon^p - \varepsilon \lambda_\varepsilon^{-\frac{q-1}{p-1}} |x|^{-(q+1)\nu} u_\varepsilon^q \quad \text{in } \Omega.$$

Using the bounds on λ_ε from Proposition 6.1, we can conclude that there exist solutions u_n of Problem (1.2) along a sequence $\{\varepsilon_n\}$ of values of ε which tends to zero as n tends to infinity. By setting $\lambda_n := \lambda_{\varepsilon_n}^{-\frac{1}{p-1}}$, theorem follows from Proposition 6.1. \square

7. THE CASE $p = 2^* - 1$ AND PROOF OF THEOREM 1.11

Lemma 7.1. *Let v_ε be as in Theorem 1.11. Then $\|v_\varepsilon\|_\infty \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.*

Proof. We have

$$(7.1) \quad \int_{\Omega} |x|^{-2^*\nu} v_\varepsilon^{2^*} dx = c,$$

where $c \in (A, B)$. If possible, let $\|v_\varepsilon\|_\infty$ be uniformly bounded. Hence by the Schauder estimate $v_\varepsilon \rightarrow v$ in $C_{loc}^2(\Omega \setminus \{0\})$, where v satisfies

$$(7.2) \quad \begin{cases} -\nabla(|x|^{-2\nu}\nabla v) = |x|^{-2^*\nu} v^{2^*-1} & \text{in } \Omega, \\ v \neq 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by the dominated convergence theorem we have

$$(7.3) \quad A < \int_{\Omega} |x|^{-2^*\nu} v^{2^*} dx < B.$$

As $A > 0$, the above expression implies v is nontrivial in a star-shaped domain which is a contradiction. \square

Define

$$(7.4) \quad \gamma_\varepsilon := \|v_\varepsilon\|_\infty^{-\frac{2}{\alpha}}.$$

Therefore $\|v_\varepsilon\|_\infty = \gamma_\varepsilon^{-\frac{\alpha}{2}}$ and $\gamma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Define

$$(7.5) \quad z_\varepsilon(x) = \gamma_\varepsilon^{\frac{\alpha}{2}} v_\varepsilon(\gamma_\varepsilon x).$$

Then $\|z_\varepsilon\|_\infty = 1$ and satisfies

$$(7.6) \quad \begin{cases} -\nabla(|x|^{-2\nu}\nabla z_\varepsilon) = |x|^{-2^*\nu} z_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} |x|^{-(q+1)\nu} z_\varepsilon^q & \text{in } \Omega_\varepsilon, \\ z_\varepsilon > 0 & \text{in } \Omega_\varepsilon, \\ z_\varepsilon = 0 & \text{in } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \gamma_\varepsilon^{-1}\Omega$.

Lemma 7.2. *Suppose z_ε is as in (7.5), $0 < \nu < \frac{N-2}{4}$, $\frac{N+2}{N-2} < q < \frac{1+\nu}{\nu}$ and (1.3) holds. Then*

$$(i) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} = 0$$

$$(ii) \quad \text{There exists } Z \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \text{ such that } z_\varepsilon \rightarrow Z \text{ in } C_{loc}^2(\mathbb{R}^N \setminus \{0\}) \cap L^\infty(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0.$$

$$(iii) \quad Z \text{ satisfies Eq. (1.22) and given by (1.20).}$$

Remark 7.1. *The upper bound of q comes from the fact that limit of $\varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}}$ can be ∞ as q is supercritical. To exclude this option we need to put this restriction on q . Note that, when q is critical or subcritical, the above limit is always 0. Therefore in the subcritical case no extra restriction on the upper bound of q appears.*

Proof. Being defined as in (7.5), z_ε satisfies Eq.(7.6). Let $\phi \in C_0^\infty(\mathbb{R}^N)$. Thus $\phi \in C_0^\infty(\Omega_\varepsilon)$ for ε small. Taking ϕ as the test function, from Eq.(7.6) we have

$$(7.7) \quad \int_{\Omega_\varepsilon} |x|^{-2\nu} \nabla z_\varepsilon \nabla \phi = \int_{\Omega_\varepsilon} |x|^{-2^* \nu} z_\varepsilon^{2^*-1} \phi - \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} z_\varepsilon^q \phi.$$

Case 1: $\varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}}$ is bounded.

Therefore there exists $c \geq 0$ such that $\varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} \rightarrow c$ (along a subsequence). Furthermore, by the elliptic regularity theory it follows that $z_\varepsilon \rightarrow Z$ in $C_{loc}^2(\mathbb{R}^N \setminus \{0\})$.

Suppose $c > 0$. Since $z_\varepsilon \rightarrow Z$ a.e and $\|z_\varepsilon\|_{L^\infty} = 1$, by dominated convergence theorem it follows

$$(7.8) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{(N+2)-q(N-2)} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} z_\varepsilon^q \phi = c \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^q \phi;$$

$$(7.9) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |x|^{-2^* \nu} z_\varepsilon^{2^*-1} \phi = \int_{\mathbb{R}^N} |x|^{-2^* \nu} Z^{2^*-1} \phi.$$

Claim: $\| |x|^{-\nu} \nabla z_\varepsilon \|_{L^2(\Omega_\varepsilon)}$ is uniformly bounded with respect to ε .

Assuming the claim,

$$(7.10) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |x|^{-2\nu} \nabla z_\varepsilon \nabla \phi = \int_{\mathbb{R}^N} |x|^{-2\nu} \nabla Z \nabla \phi,$$

follows from Vitali's convergence theorem, since $\nabla z_\varepsilon \rightarrow \nabla Z$ a.e. in \mathbb{R}^N .

To prove the claim, we see

$$(7.11) \quad \begin{aligned} \int_{\Omega_\varepsilon} |x|^{-2\nu} |\nabla z_\varepsilon|^2 &= \int_{\Omega_\varepsilon} |x|^{-2^* \nu} z_\varepsilon^{2^*} - \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} z_\varepsilon^{q+1} \\ &\leq \int_{\Omega} |x|^{-2^* \nu} v_\varepsilon^{2^*} = 1. \end{aligned}$$

Combining (7.8)-(7.10), we have

$$(7.12) \quad -\operatorname{div}(|x|^{-2\nu} \nabla Z) = |x|^{-2^* \nu} Z^{2^*-1} - c |x|^{-(q+1)\nu} Z^q \quad \text{in } \mathbb{R}^N.$$

Moreover, by Fatou's lemma

$$(7.13) \quad \begin{aligned} c \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx &\leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} z_\varepsilon^{q+1} dx \\ &= \liminf_{\varepsilon \rightarrow 0} \left[\int_{\Omega_\varepsilon} |x|^{-2^* \nu} z_\varepsilon^{2^*} dx - \int_{\Omega_\varepsilon} |x|^{-2\nu} |\nabla z_\varepsilon|^2 dx \right] \\ &\leq 1. \end{aligned}$$

Since $c > 0$ and $z_\varepsilon \rightarrow Z$ in $C_{loc}^2(\mathbb{R}^N \setminus \{0\})$, from (7.11) and (7.13), it follows that $Z \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu})$. Therefore using Pohozaev identity (see (4.13)), we have

$$c \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx = 0,$$

which is a contradiction as $|Z|_{L^\infty} = 1$. Therefore, $c = 0$. Consequently, (7.12) yields Z satisfies (1.22).

Case 2: $\lim_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} = \infty$.

Set, $\lambda_\varepsilon := \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}}$ and define $\tilde{z}_\varepsilon(x) := z_\varepsilon(\frac{x}{\lambda_\varepsilon^m})$, where $m = \frac{1}{2+(q-1)\nu}$.

A straight forward computation yields, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$(7.14) \quad \begin{aligned} \int_{\lambda_\varepsilon^m \Omega_\varepsilon} |x|^{-2\nu} \nabla \tilde{z}_\varepsilon(x) \nabla \psi(x) dx &= \lambda_\varepsilon^{\frac{2^*\nu-2\nu-2}{2+(q-1)\nu}} \int_{\lambda_\varepsilon^m \Omega_\varepsilon} |x|^{-2^*\nu} \tilde{z}_\varepsilon^{2^*-1}(x) \psi(x) dx \\ &- \int_{\lambda_\varepsilon^m \Omega_\varepsilon} |x|^{-(q+1)\nu} \tilde{z}_\varepsilon^q(x) \psi(x) dx. \end{aligned}$$

Since $\lambda_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$, we obtain $\lambda_\varepsilon^m \Omega_\varepsilon \rightarrow \mathbb{R}^N$ and $\lambda_\varepsilon^{\frac{2^*\nu-2\nu-2}{2+(q-1)\nu}} \rightarrow 0$. Using elliptic regularity theory we can argue as before that there exists \tilde{Z} such that $\tilde{z}_\varepsilon \rightarrow \tilde{Z}$ in $C_{loc}^2(\mathbb{R}^N \setminus \{0\})$. Moreover, $\|z_\varepsilon\|_{L^\infty} = 1$ implies $\|\tilde{z}_\varepsilon\|_{L^\infty} = 1$. Therefore arguing as in Case 1, we can prove that \tilde{Z} satisfies the following equation:

$$(7.15) \quad -\operatorname{div}(|x|^{-2\nu} \nabla \tilde{Z}) + |x|^{-(q+1)\nu} \tilde{Z}^q = 0 \quad \text{in } \mathbb{R}^N.$$

From Theorem A.1 (see Appendix A), it follows that $\tilde{Z} = 0$. This is a contradiction as $\|\tilde{z}_\varepsilon\|_{L^\infty} = 1$ implies $\|\tilde{Z}\|_{L^\infty} = 1$. Hence Case 2 can not occur. Therefore from Case 1 we conclude (i) holds and $z_\varepsilon \rightarrow Z$ in $C_{loc}^2(\mathbb{R}^N \setminus \{0\})$.

Since Z satisfies (1.22), Z must be of the form $\mu^{-\frac{\alpha}{2}} U(\frac{x}{\mu})$, where U is as in (1.20) for some $\mu > 0$. By (1.3), it follows that $\max z_\varepsilon = z_\varepsilon(0) = 1$. This implies $Z(0) = 1$ and $0 \leq Z \leq 1$. From this it follows $z_\varepsilon \rightarrow Z$ in $L_{loc}^\infty(\mathbb{R}^N)$ and $\mu = \left(\alpha \sqrt{\frac{N}{N-2}} \right)^{\frac{N-2}{\alpha}}$.

From this, direct calculation yields that $Z(x) = \left(1 + \frac{|x|^{\frac{2\alpha}{N-2}}}{\frac{N\alpha^2}{N-2}} \right)^{-\frac{N-2}{2}}$. \square

We know the local behavior of z_ε . Now we need to check the behavior of z_ε near ∞ . Hence define the Kelvin transform of z_ε as

$$(7.16) \quad \hat{z}_\varepsilon(x) = |x|^{-\alpha} z_\varepsilon\left(\frac{x}{|x|^2}\right) \text{ in } \Omega_\varepsilon \setminus \{0\}.$$

Then from (7.6), \hat{z}_ε satisfies

$$(7.17) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla \hat{z}_\varepsilon) = |x|^{-2^*\nu} \hat{z}_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} |x|^{-(q+1)\nu+\alpha(q-2^*+1)} \hat{z}_\varepsilon^q & \text{in } \Omega_\varepsilon^* \\ \hat{z}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon^*. \end{cases}$$

where Ω_ε^* is the image Ω_ε under the Kelvin transform. Hence the behavior of z_ε near ∞ amounts to behavior of \hat{z}_ε near 0.

Lemma 7.3. *There exist $R > 0$ and $C > 0$ independent of $\varepsilon > 0$ such that any solution of (7.17) satisfy*

$$(7.18) \quad \|\hat{z}_\varepsilon\|_{L^\infty(B_R)} \leq C \left(\int_{B_R} |x|^{-2^*\nu} \hat{z}_\varepsilon^{2^*} dx \right)^{\frac{1}{2^*}}.$$

Proof. The proof of the above lemma follows along the same line of arguments as in Theorem 1.4 (i) with a suitable modification and we skip the proof. \square

Remark 7.2. *There exists $C > 0$ independent of $\varepsilon > 0$ such that $z_\varepsilon \leq CZ(x)$ for all $x \in \Omega_\varepsilon$. For this, note that $\|z_\varepsilon\|_\infty = 1$, this implies that $z_\varepsilon \leq CZ(x)$ locally.*

From (7.3) we have

$$A < \int_{\Omega_\varepsilon} |x|^{-2^*\nu} z_\varepsilon^{2^*} dx < B.$$

But this implies that

$$\int_{B_R} |x|^{-2^*\nu} \hat{z}_\varepsilon^{2^*} dx \leq \int_{\Omega_\varepsilon} |x|^{-2^*\nu} z_\varepsilon^{2^*} dx < B$$

and since at infinity Z decays as $|x|^{-\alpha}$, we have $z_\varepsilon \leq CZ(x)$ near infinity. Hence, we have $z_\varepsilon \leq CZ(x)$ for all $x \in \Omega_\varepsilon$. As a conclusion, from (7.5) we obtain that there exists $C > 0$ independent of ε such that

$$(7.19) \quad v_\varepsilon(x) \leq C\gamma_\varepsilon^{-\frac{\alpha}{2}} Z\left(\frac{x}{\gamma_\varepsilon}\right).$$

Define $w_\varepsilon(x) = \|v_\varepsilon\|_\infty v_\varepsilon(x) = \gamma_\varepsilon^{-\frac{\alpha}{2}} v_\varepsilon(x)$. Then w_ε satisfies

$$(7.20) \quad \begin{cases} -\operatorname{div}(|x|^{-2\nu} \nabla w_\varepsilon) = \gamma_\varepsilon^{-\frac{\alpha}{2}} |x|^{-2^*\nu} v_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} |x|^{-(q+1)\nu} v_\varepsilon^q & \text{in } \Omega \\ w_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 7.4. *Let ν and q be as in Lemma 7.2 and w_ε be as in (7.20). Then there exists a constant $C > 0$ such that*

$$\|w_\varepsilon\|_{L^\infty(K)} + \|\nabla w_\varepsilon\|_{L^\infty(K)} \leq C,$$

for every compact subset K of $\Omega \setminus \{0\}$.

Proof. Using the Green kernel's representation and Lemma 4.2, we have

$$\begin{aligned} |w_\varepsilon(x)| &= \gamma_\varepsilon^{-\frac{\alpha}{2}} \left| \int_{\Omega} G(x, y) [|y|^{-2^*\nu} v_\varepsilon^{2^*-1} - \varepsilon |y|^{-(q+1)\nu} v_\varepsilon^q] dy \right| \\ &\leq C\gamma_\varepsilon^{-\frac{\alpha}{2}} |x|^{2\nu} \int_{\Omega} |x-y|^{2-N} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy \\ &\quad + C\gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega} |x-y|^{2+2\nu-N} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy \\ &\quad + C\varepsilon\gamma_\varepsilon^{-\frac{\alpha}{2}} |x|^{2\nu} \int_{\Omega} |x-y|^{2-N} |y|^{-(q+1)\nu} v_\varepsilon^q dy \\ &\quad + C\varepsilon\gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega} |x-y|^{2+2\nu-N} |y|^{-(q+1)\nu} v_\varepsilon^q dy \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Moreover,

$$\begin{aligned} I_1 &:= C|x|^{2\nu} \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega} |x-y|^{2-N} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy \\ &= C|x|^{2\nu} \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \cap B_{\frac{|x|}{2}}(0)} |x-y|^{2-N} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy \\ &\quad + C|x|^{2\nu} \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \setminus B_{\frac{|x|}{2}}(0)} |x-y|^{2-N} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy \\ &=: I_{11} + I_{12}. \end{aligned}$$

Using (7.19) along with the facts that $Z(x) \sim |x|^{-\alpha}$ at infinity and $\gamma_\varepsilon \rightarrow 0$, we find

$$(7.21) \quad \gamma_\varepsilon^{-\frac{\alpha}{2}} |y|^{-2^*\nu} v_\varepsilon^{2^*-1}(y) \leq \frac{C}{|y|^{(N+2)-\frac{4\nu}{N-2}}} \quad \text{in } \Omega \setminus B_{\frac{|x|}{2}}(0),$$

$$(7.22) \quad \gamma_\varepsilon^{-\frac{\alpha}{2}} |y|^{-(q+1)\nu} v_\varepsilon^q(y) \leq \frac{C\gamma_\varepsilon^{(q-1)\frac{\alpha}{2}}}{|y|^{(N-2)q-\nu(q-1)}} \quad \text{in } \Omega \setminus B_{\frac{|x|}{2}}(0).$$

Hence

$$\begin{aligned} I_{12} &\leq C|x|^{2\nu} \int_{\Omega \setminus B_{\frac{|x|}{2}}(0)} \frac{1}{|x-y|^{N-2}|y|^{(N+2)-\frac{4\nu}{N-2}}} dy \\ &\leq \frac{C|x|^{2\nu}}{|x|^{(N+2)-\frac{4\nu}{N-2}}} \int_{\Omega \setminus B_{\frac{|x|}{2}}(0)} |x-y|^{2-N} dy. \end{aligned}$$

When $y \in \Omega \cap B_{\frac{|x|}{2}}(0)$, we have $|x-y| \geq |x|-|y| \geq \frac{1}{2}|x|$. Therefore using (7.19), we get

$$\begin{aligned} I_{11} &\leq \frac{C|x|^{2\nu}\gamma_\varepsilon^{-\frac{\alpha}{2}}}{|x|^{N-2}} \int_{\Omega \cap B_{\frac{|x|}{2}}(0)} |y|^{-2^*\nu} v_\varepsilon^{2^*-1}(y) dy \\ &\leq \frac{C\gamma_\varepsilon^{-\frac{2^*\alpha}{2}}}{|x|^\alpha} \int_{\Omega \cap B_{\frac{|x|}{2}}(0)} |y|^{-2^*\nu} Z\left(\frac{y}{\gamma_\varepsilon}\right)^{2^*-1} dy \\ &\leq \frac{C}{|x|^\alpha} \gamma_\varepsilon^{-\frac{2^*\alpha}{2}-2^*\nu+N} \int_{\mathbb{R}^N} |y|^{-2^*\nu} Z(y)^{2^*-1} dy \\ &= \frac{C}{|x|^\alpha} \int_{\mathbb{R}^N} |y|^{-2^*\nu} Z(y)^{2^*-1} dy \\ (7.23) \quad &= \frac{C}{|x|^\alpha} \omega_N \alpha^{N-1} \left(\frac{N}{N-2}\right)^{\frac{N-2}{2}}, \end{aligned}$$

where the last integral can be computed as in (7.32) in Lemma 7.6. Similarly I_2 can be written as

$$\begin{aligned} I_2 &= C\gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \cap B_{\frac{|x|}{2}}(0)} |x-y|^{2+\nu-N} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy \\ &\quad + C\gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \setminus B_{\frac{|x|}{2}}(0)} |x-y|^{2+\nu-N} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy \\ (7.24) \quad &=: I_{21} + I_{22}. \end{aligned}$$

Proceeding similarly as we did for I_{12} and I_{11} , we have

$$\begin{aligned} I_{22} &\leq C \int_{\Omega \setminus B_{\frac{|x|}{2}}(0)} \frac{1}{|x-y|^{N-2\nu-2}|y|^{(N+2)-\frac{4\nu}{N-2}}} dy \\ &\leq \frac{C}{|x|^{(N+2)-\frac{4\nu}{N-2}}} \int_{\Omega \setminus B_{\frac{|x|}{2}}(0)} |x-y|^{2+2\nu-N} dy; \\ (7.25) \quad I_{21} &\leq \frac{C}{|x|^\alpha} \omega_N \alpha^{N-1} \left(\frac{N}{N-2}\right)^{\frac{N-2}{2}}. \end{aligned}$$

Similarly to compute I_3 and I_4 , we break those integral into two parts, namely in $\Omega \cap B_{\frac{|x|}{2}}(0)$ and $\Omega \setminus B_{\frac{|x|}{2}}(0)$. Using (7.22), integral in $\Omega \setminus B_{\frac{|x|}{2}}(0)$ can be computed as before. Proceeding as in (7.23), we have

$$\begin{aligned} & \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} |x|^{2\nu} \int_{\Omega \cap B_{\frac{|x|}{2}}(0)} |x-y|^{2-N} |y|^{-(q+1)\nu} v_\varepsilon^q dy \\ & \leq \frac{C \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}}}{|x|^\alpha} \int_{\Omega \cap B_{\frac{|x|}{2}}(0)} |y|^{-(q+1)\nu} v_\varepsilon^q(y) dy \\ & \leq \frac{C \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}}}{|x|^\alpha} \int_{\mathbb{R}^N} |y|^{-(q+1)\nu} Z^q(y) dy \end{aligned}$$

By a straight forward computation using the expression of Z from Lemma 7.2, it can shown that $\int_{\mathbb{R}^N} |y|^{-(q+1)\nu} Z^q(y) dy < \infty$. Moreover, as $\varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} \rightarrow 0$ (see Lemma 7.2), we can conclude that for any compact set $K \subset \Omega \setminus \{0\}$, we have $\|w_\varepsilon\|_{L^\infty(K)} \leq C$ and by the regularity $\|\nabla w_\varepsilon\|_{L^\infty(K)} \leq C$. \square

Lemma 7.5. *Let ν, q, w_ε be as in Lemma 7.4. Then there exists $\gamma_0 > 0$ such that*

$$(7.26) \quad \lim_{\varepsilon \rightarrow 0} w_\varepsilon(x) = \gamma_0 G(x, 0) \text{ in } C_{loc}^1(\bar{\Omega} \setminus \{0\}).$$

Proof. Define

$$f_\varepsilon := \gamma_\varepsilon^{-\frac{\alpha}{2}} |x|^{-2^* \nu} v_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} |x|^{-(q+1)\nu} v_\varepsilon^q.$$

Choose $R > 0$ such that $\Omega' = \Omega \setminus \overline{B_R}(0)$ is connected. Then $|w_\varepsilon| + |\nabla w_\varepsilon| \leq C$ for all $x \in \Omega'$. Let $x' \in \partial\Omega \cap \partial\Omega'$, then $|w_\varepsilon(x) - w_\varepsilon(x')| \leq C$ for all $x \in \Omega'$. But this implies w_ε is uniformly bounded in $\Omega' \cap \bar{\Omega}$. By the standard regularity, we have $w_\varepsilon \rightarrow w$ as $\varepsilon \rightarrow 0$ in $C_{loc}^1(\bar{\Omega} \setminus \{0\})$. If $K \subseteq \bar{\Omega} \setminus \{0\}$, then for any $x \in K$ and $r > 0$ small, using the fact $\gamma_\varepsilon \rightarrow 0$, we have

$$\begin{aligned} w_\varepsilon(x) &= \int_{\Omega} G(x, y) f_\varepsilon(y) dy \\ &= \int_{B_r(0)} G(x, y) f_\varepsilon(y) dy + \int_{\Omega \setminus B_r(0)} G(x, y) f_\varepsilon(y) dy \\ &= \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{B_r(0)} G(x, y) |y|^{-2^* \nu} v_\varepsilon^{2^*-1}(y) dy \\ &\quad + \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-2^* \nu} v_\varepsilon^{2^*-1}(y) dy \\ &\quad + \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_\varepsilon^q(y) dy \\ (7.27) \quad &\quad + \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_\varepsilon^q(y) dy \end{aligned}$$

Claim: (i) $\varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_\varepsilon^q(y) dy = o(1)$ and

(ii) $\gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-2^* \nu} v_\varepsilon^{2^*-1}(y) dy = o(1)$.

To see this,

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_\varepsilon^q(y) dy \\
& \leq \lim_{\varepsilon \rightarrow 0} C \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}(q+1)} \int_{\Omega \setminus B_r(0)} [|x|^{2\nu} |x-y|^{2-N} + |x-y|^{-\alpha}] |y|^{-(q+1)\nu} Z^q\left(\frac{y}{\gamma_\varepsilon}\right) dy \\
& \leq \lim_{\varepsilon \rightarrow 0} C \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}(q+1)} \int_{\Omega \setminus B_r(0)} [|x|^{2\nu} |x-y|^{2-N} + |x-y|^{-\alpha}] |y|^{-(q+1)\nu} \left|\frac{y}{\gamma_\varepsilon}\right|^{-\alpha q} dy \\
& \leq \lim_{\varepsilon \rightarrow 0} C \varepsilon \gamma_\varepsilon^{\frac{\alpha}{2}(q-1)} \int_{\Omega \setminus B_r(0)} [|x|^{2\nu} |x-y|^{2-N} + |x-y|^{-\alpha}] |y|^{-(q+1)\nu-\alpha q} dy \\
& = o(1).
\end{aligned}$$

Similarly (ii) follows. Therefore from (7.27), we obtain

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} w_\varepsilon(x) &= \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{B_r(0)} G(x, y) |y|^{-2^*\nu} v_\varepsilon^{2^*-1}(y) dy \\
&+ \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_\varepsilon^q(y) dy + o(1).
\end{aligned}$$

Furthermore $G(x, \cdot)$ is continuous in $\overline{\Omega} \setminus \{x\}$, we obtain

$$(7.28) \quad w_\varepsilon(x) = \gamma_\varepsilon^{-\frac{\alpha}{2}} G(x, 0) \int_{B_r(0)} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy + L + o(1).$$

where

$$L = \varepsilon \gamma_\varepsilon^{-\frac{\alpha}{2}} G(x, 0) \int_{B_r(0)} |y|^{-(q+1)\nu} v_\varepsilon^q dy.$$

By doing a straight forward computation using (7.19), it follows

$$L \leq \varepsilon \gamma_\varepsilon^{\frac{(N+2)-q(N-2)}{2}} \int_{\mathbb{R}^N} |y|^{-(q+1)\nu} Z^q(y) dy.$$

Consequently, a direct computation using Lemma 7.2 yields $L = o(1)$.

Define

$$\gamma_0 := \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{B_r(0)} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy.$$

Then

$$(7.29) \quad \lim_{\varepsilon \rightarrow 0} w_\varepsilon(x) = \gamma_0 G(x, 0).$$

Moreover from Lemma 7.6 we get $\gamma_0 = \omega_N (N-2-2\nu)^{N-1} \left(\frac{N}{N-2}\right)^{\frac{N-2}{2}}$.

Using the same procedure as above we can show that

$$w_\varepsilon \rightarrow \gamma_0 G(x, 0) \text{ in } C_{loc}^1(\overline{\Omega} \setminus \{0\}).$$

□

Lemma 7.6. *Let v_ε be as in Theorem 1.11 and γ_ε be as defined in (7.4). Define*

$$\gamma_0 := \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{B_r(0)} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy. \text{ Then}$$

$$(7.30) \quad \gamma_0 = \omega_N (N-2-2\nu)^{N-1} \left(\frac{N}{N-2}\right)^{\frac{N-2}{2}}.$$

Proof. We define $I_{\varepsilon,r} := \gamma_\varepsilon^{-\frac{\alpha}{2}} \int_{B_r(0)} |y|^{-2^*\nu} v_\varepsilon^{2^*-1} dy$. Since v_ε and z_ε are related by

$$(7.5), \text{ we have } v_\varepsilon(x) = \gamma_\varepsilon^{-\frac{\alpha}{2}} z_\varepsilon\left(\frac{x}{\gamma_\varepsilon}\right). \text{ Thus}$$

$$(7.31) \quad I_{\varepsilon,r} = \gamma_\varepsilon^{-\frac{\alpha}{2}-2^*\nu-\frac{\alpha}{2}(2^*-1)+N} \int_{\frac{B_r(0)}{\gamma_\varepsilon}} |x|^{-2^*\nu} z_\varepsilon^{2^*-1}(x) dx = \int_{\frac{B_r(0)}{\gamma_\varepsilon}} |x|^{-2^*\nu} z_\varepsilon^{2^*-1}(x) dx$$

Since $\varepsilon \rightarrow 0$ implies $\gamma_\varepsilon \rightarrow 0$, we have

$$(7.32) \quad \gamma_0 = \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_{\varepsilon,r} = \int_{\mathbb{R}^N} |x|^{-2^*\nu} Z^{2^*-1} dx,$$

where Z is as in Lemma 7.2. Therefore by doing a straight forward computation, we obtain

$$\gamma_0 = \frac{\omega_N N \alpha}{2} \left(\frac{N \alpha^2}{N-2} \right)^{\frac{N-2}{2}} B\left(\frac{N}{2}, 1\right),$$

where $B(a, b) = \int_0^\infty t^{a-1} (1+t)^{-a-b} dt$ is the Beta function. Recall that $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Thus $B\left(\frac{N}{2}, 1\right) = \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2}+1)} = \frac{\Gamma(\frac{N}{2})}{\frac{N}{2}\Gamma(\frac{N}{2})} = \frac{2}{N}$, the lemma follows. \square

Proof of Theorem 1.11 . From (4.13) we have

$$\frac{1}{2} \int_{\partial\Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla v_\varepsilon|^2 dS = \varepsilon \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\Omega} |x|^{-(q+1)\nu} v_\varepsilon^{q+1} dx.$$

Using $w_\varepsilon = \|v_\varepsilon\|_\infty v_\varepsilon$ in the above expression, we have

$$\begin{aligned} \int_{\partial\Omega} |x|^{-2\nu} |\nabla w_\varepsilon|^2 \langle x, n \rangle dS &= 2\varepsilon \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \|v_\varepsilon\|_\infty^2 \int_{\Omega} |x|^{-(q+1)\nu} v_\varepsilon^{q+1} dx \\ &= 2\varepsilon \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \|v_\varepsilon\|_\infty^{\frac{q(N-2)-(N+2)+2\alpha}{\alpha}} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} z_\varepsilon^{q+1} dx. \end{aligned}$$

Since $z_\varepsilon \rightarrow Z$ a.e and $z_\varepsilon \leq CZ$, by the dominated convergence theorem it follows $\int_{\Omega_\varepsilon} |x|^{-(q+1)\nu} z_\varepsilon^{q+1} dx \rightarrow \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx$. Therefore taking limit $\varepsilon \rightarrow 0$ and using (7.26) and Lemma 4.5, we obtain

$$(7.33) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \|v_\varepsilon\|_\infty^{\frac{q(N-2)-(N+2)+2\alpha}{\alpha}} = \frac{(N-2-2\nu)\gamma_0^2 |R(0)|}{2 \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx}.$$

From Lemma 7.2, we know $Z(x) = \left(1 + \frac{|x|^{\frac{2\alpha}{N-2}}}{\frac{N\alpha^2}{N-2}} \right)^{-\frac{N-2}{2}}$. Therefore a straight forward calculation yields

$$(7.34) \quad \begin{aligned} \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx &= \frac{\omega_N N \alpha}{2} \left(\frac{N \alpha^2}{N-2} \right)^{(N-(q+1)\nu)\frac{N-2}{2\alpha}-1} \\ &\times B\left(\frac{N-2}{2\alpha}(N-(q+1)\nu), \frac{N-2}{2\alpha}\{q(N-2-\nu)-(2+\nu)\}\right), \end{aligned}$$

where $B(a, b) = \int_0^\infty t^{a-1}(1+t)^{-a-b}dt$ and ω_N is the surface measure of unit sphere in \mathbb{R}^N . From Lemma 7.6, it is known that $\gamma_0 = \omega_N \alpha^{N-1} \left(\frac{N}{N-2} \right)^{\frac{N-2}{2}}$. Substituting the value of γ_0 , α , and $\int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx$ in (7.33), we have RHS of (7.33) as

$$\begin{aligned} RHS &= \frac{\alpha(q+1)|R(0)|}{\{(N-2)q-(N+2)\}} \cdot \frac{\omega_N^2 \alpha^{2(N-1)} \left(\frac{N}{N-2} \right)^{N-2}}{\omega_N \frac{N\alpha}{2} \left(\frac{N\alpha^2}{N-2} \right)^{(N-(q+1)\nu) \left(\frac{N-2}{2\alpha} \right)^{-1}}} \times \\ &\quad \left[B \left(\frac{N-2}{2\alpha} (N - (q+1)\nu), \frac{N-2}{2\alpha} \{q(N-2-\nu) - (2+\nu)\} \right) \right]^{-1} \\ &= \frac{\omega_N |R(0)|}{C_{q,N}} \frac{(N-2-2\nu) \frac{(N-(q+1)\nu)(N-2-4N\nu)}{\alpha} (N-2) \frac{(N-(q+1)\nu)(N-2-2\alpha(N-1))}{2\alpha}}{N \frac{(N-(q+1)\nu-2\alpha)(N-2)}{2\alpha}} \\ &\quad \times \left[B \left(\frac{N-2}{2\alpha} (N - (q+1)\nu), \frac{N-2}{2\alpha} \{q(N-2-\nu) - (2+\nu)\} \right) \right]^{-1}, \end{aligned}$$

where $C_{q,N}$ is as in (1.37).

Furthermore, (1.38) follows from (7.26). \square

APPENDIX A.

In this section we consider the following problem:

$$(1.1) \quad \begin{aligned} -\operatorname{div}(|x|^{-2\nu} \nabla u) + |x|^{-(q+1)\nu} u^q &= 0 \quad \text{in } \mathbb{R}^N, \\ u &\geq 0 \quad \text{in } \mathbb{R}^N, \end{aligned}$$

where $\frac{N+2}{N-2} < q < \frac{1+\nu}{\nu}$ and $0 < \nu < \frac{N-2}{4}$.

Definition A.1. If u is a solution of (1.1) in a domain Ω such that $u(x) \rightarrow \infty$ as $x \rightarrow \partial\Omega$, then u is called a large solution.

Theorem A.1. Suppose u is any solution of Eq.(1.1). Then $u = 0$.

Proof. Let U_1 be a large solution of (1.1) in $B_1(0)$ such that $U_1 \in L_{loc}^\infty(B_1(0))$ (existence of such solution follows from Theorem A.2). Define

$$U_R(x) := R^{\nu - \frac{2}{q-1}} U_1\left(\frac{x}{R}\right).$$

By a straight forward computation, it follows U_R is a large solution of (1.1) in $B_R(0)$. Moreover, $U_R \rightarrow 0$ as $R \rightarrow \infty$. If u is any solution of (1.1) in \mathbb{R}^N , then u is a solution of (1.1) in $B_R(0)$ as well. Consequently,

$$-\operatorname{div}(|x|^{-2\nu} \nabla(u - U_R)) + |x|^{-(q+1)\nu} (u^q - U_R^q) = 0 \quad \text{in } B_R(0).$$

Clearly $u \leq U_R$ on $\partial B_R(0)$. Therefore by taking $(u - U_R)^+$ as a test function for the above equation we obtain

$$\int_{B_R(0)} |x|^{-2\nu} |\nabla(u - U_R)^+|^2 + \int_{B_R(0)} |x|^{-(q+1)\nu} \frac{(u^q - U_R^q)}{u - U_R} |(u - U_R)^+|^2 = 0,$$

which in turn implies $(u - U_R)^+ = 0$ in $B_R(0)$. Thus $u \leq U_R$ in $B_R(0)$. Hence by taking limit $R \rightarrow \infty$, we have $u \leq 0$ in \mathbb{R}^N . Since u is a nonnegative solution, we get $u = 0$ in \mathbb{R}^N . \square

Theorem A.2. *There exists a large solution u to the equation (1.1) in $B_1(0)$. Moreover, $u \in L_{loc}^\infty(B_1(0))$.*

We essentially follow the classical method of Veron-Vazquez [24, Lemma 2.1] to prove this result.

Proof. We will show that there exists a radial large solution. Towards this goal, let us consider the following ode

$$(1.2) \quad \begin{cases} u'' + \frac{N-1-2\nu}{r}u'(r) = r^{-(q-1)\nu}u^q & \text{in } (0, 1) \\ u > 0 & \text{in } (0, 1) \\ u(0) = 1 \quad u'(0) = 0. \end{cases}$$

Then we can write the solution as

$$u(r) = 1 + \int_0^r s^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} u^q(t) dt ds.$$

Since $q < \frac{1+\nu}{\nu}$ implies $q < \frac{2+\nu}{\nu} < \frac{N-\nu}{\nu}$, by the standard existence of ode theory, it follows that solution $u(r)$ exists in a neighborhood of 0. Moreover, $q < \frac{1+\nu}{\nu}$ implies $u'(0) = 0$ and u is C^1 up to the blow-up time.

Claim: There exists a solution u of the following ode

$$u'' + \frac{N-1-2\nu}{r}u'(r) = r^{-(q-1)\nu}u^q$$

in $[0, r^*)$ such that $\lim_{r \uparrow r^*} u(r) = +\infty$ for some $r^* > 0$.

To see the claim, we use generalised Emden–Fowler transform $t = (\frac{\alpha}{r})^\alpha$ and $y(t) = \alpha^{-\nu}u(r)$, where $\alpha = N - 2 - 2\nu$. Therefore we obtain

$$(1.3) \quad y''(t) = t^{\frac{-(2\alpha+2)+(q-1)\nu}{\alpha}} y^q \quad \text{in } R < t < +\infty.$$

Existence of $u(r)$ in the neighbourhood of 0 implies, Eq. (1.3) has a solution $y(t)$ in (R, ∞) for some large $R > 0$ with $y'(\infty) = 0$, $y(\infty) > 0$. To prove the claim, it is equivalent to show that there exists a solution y of (1.3) in (t^*, ∞) such that $\lim_{t \downarrow t^*} y(t) = \infty$ for some $t^* \in (0, \infty)$. Suppose this is not true, then $y(t)$ can be continued as a solution of (1.3) to the left of ∞ till 0, i.e $y(t)$ can be defined on $(0, \infty)$.

Set $f(t) := t^{\frac{-(2\alpha+2)+(q-1)\nu}{\alpha}}$ and let $0 < R < R' < \infty$. As f is continuous and positive we get there exists $m, M > 0$ such that $0 < m \leq f(t) \leq M$ for $t \in [R, R']$. Now consider the ode

$$(1.4) \quad v''(t) = Mv^q(t) \quad \text{in } (R, R'); \quad v > 0 \quad \text{in } (R, R').$$

Rename the nonlinear term in (1.4) as $h(v)$, that is $h(t) := Mt^q$. Then

$$H(t) := \int_0^t h(s)ds, \quad \psi(a) := \int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty,$$

for any $a > 0$. Therefore, applying Vazquez's classical a-priori estimates [23] (also see [24, Lemma 2.1]) we find a large solution $v(t)$ of (1.4). That is, $\lim_{t \downarrow R} v(t) =$

$\infty = \lim_{t \uparrow R'} v(t)$. Using comparison principle it is easy to check that any solution y of (1.3) satisfies

$$(1.5) \quad y(t) \leq v(t) \quad \text{in } (R, R').$$

From (1.4), it also follows that v is a convex function. If y is a solution of (1.3) in (T, ∞) for some large T with the initial value $y(\infty) > \min_{R < t < R'} v(t)$ and $y'(\infty) = 0$, then graph of y must lie above all of its tangent as y is a convex decreasing function. Consequently, $y(t) > \min_{R < s < R'} v(s)$ for all $t < \infty$. Since y can be extended till 0, it in turn implies, there exists t_1, t_2 such that $R < t_1 < t_2 < R'$ and $y(t) > v(t)$ in (t_1, t_2) . This is a contradiction to (1.5). Hence y can not be defined to the left of ∞ till R , that is, there must exist $t^* > R$ such that $\lim_{t \downarrow t^*} y(t) = \infty$. This proves the claim. Since we have proved existence of a large solution u of (1.1) in the ball $B_r(0)$, we use similarity transformation T_r to get large solution in the unit ball $B_1(0)$. More precisely, $U_1(x) := T_r u(x) := r^{\frac{2}{q-1}-\nu} u(rx)$. This completes the proof of the theorem. \square

APPENDIX B.

Lemma B.1. Define $w(B_t(x)) := \int_{|x-y|<t} |y|^{-2\nu} dy$. Then

$$(2.1) \quad w(B_{2^k r}(x)) \geq C 2^{k(N-2\nu)} w(B_r(x))$$

Proof. We prove the lemma considering into three cases.

Case 1: Suppose $r \geq \frac{|x|}{10}$.

Then $2^k r \geq \frac{|x|}{10}$. We claim $B_{2^k r}(0) \subset B_{2^k \cdot 10r}(x)$. Indeed, $y \in B_{2^k r}(0)$ implies $|y - x| \leq |y| + |x| \leq (2^k r + 10r) \leq (2^k \cdot 10)r$. Thus the claim follows. Therefore using doubling measure property of $|y|^{-2\nu}$, we get

$$w(B_{2^k r}(x)) \geq c_1 w(B_{2^k \cdot 10r}(x)) \geq c_1 w(B_{2^k r}(0)),$$

where c_1 does not depend on k, r, x . As a consequence,

$$w(B_{2^k r}(x)) \geq c_1 \omega_N (2^k r)^{N-2\nu} = c_1 2^{k(N-2\nu)} w(B_r(0)) \geq c_1 2^{k(N-2\nu)} w(B_r(x)).$$

Case 2: $r < \frac{|x|}{10}$ and $2^k r < \frac{|x|}{10}$.

Then $y \in B_{2^k r}(x)$ implies

$$|x| \leq |x - y| + |y| \leq 2^k r + |y| \leq \frac{|x|}{10} + |y| \implies \frac{9}{10}|x| \leq |y|.$$

Similarly it follows $|y| \leq \frac{11}{10}|x|$. Thus $c_1|x| \leq |y| \leq c_2|x|$. If $y \in B_r(x)$, then using $r < \frac{|x|}{10}$ and doing the calculation as above we get there exists positive constants c_3, c_4 , independent of r, x, k such that $c_3|x| \leq |y| \leq c_4|x|$. Consequently,

$$w(B_{2^k r}(x)) \geq \omega_N c_2^{-2\nu} |x|^{-2\nu} (2^k r)^N \geq \omega_N c_2^{-2\nu} 2^{k(N-2\nu)} |x|^{-2\nu} r^N$$

Moreover,

$$w(B_r(x)) \leq \omega_N c_3^{-2\nu} |x|^{-2\nu} r^N.$$

Hence (2.1) holds with $C = (\frac{c_2}{c_3})^{-2\nu}$.

Case 3: $r < \frac{|x|}{10}$ and $2^k r \geq \frac{|x|}{10}$.

This case is similar to Case 1 and we skip the proof. \square

Acknowledgement The results in Appendix A are based on the ideas given by Prof Laurent Véron. The authors are indebted to him for his suggestions and also for providing the references [23] and [24]. The authors also would like to thank Dr. Anup Biswas for the idea of the proof of Theorem 1.3.

The first author is supported by INSPIRE research grant DST/INSPIRE 04/2013/000152 and the second author acknowledges funding from LMAP UMR CNRS 5142, Université Pau et des Pays de l'Adour.

REFERENCES

- [1] R. BELLMAN, Stability theory of differential equations, *International Series in Pure and Applied Math.*, Mc Graw-Hill (1953).
- [2] H. BREZIS; L. A. PELETIER, Asymptotics for elliptic equations involving critical growth. Partial differential equations and the calculus of variations, Vol. I, 149–192, *Progr. Nonlinear Differential Equations Appl.*, 1, Birkhäuser Boston, Boston, MA, 1989.
- [3] L. CAFFARELLI; R. KOHN; L. NIRENBERG, First Order Interpolation Inequalities with Weight, *Compositio Math.* **53** (1984), 259–275.
- [4] D. CAO; S. PENG, Asymptotic behavior for elliptic problems with singular coefficient and nearly critical Sobolev growth. *Ann. Mat. Pura Appl.* (4) **185** (2006), no. 2, 189–205.
- [5] F. CATRINA; Z-Q. WANG, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.* **54** (2001), no. 2, 229–258.
- [6] S. CHANILLO; R. WHEEDEN, Existence and estimates of Green's function for degenerate elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **15** (1988), no. 2, 309–340.
- [7] S. CHANILLO; R. WHEEDEN, Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations. *Comm. Partial Differential Equations* (1986), no. 10, 1111–1134.
- [8] E. EGNELL, Elliptic boundary value problems with singular coefficients and critical nonlinearities, *Indiana Univ. Math. J.* **38** (1989) 235–251.
- [9] A. FERRERO; F. GAZZOLA, Existence of solutions for singular critical growth semilinear elliptic equations. *J. Differential Equations* **177** (2001), no. 2, 494–522.
- [10] R. H. FOWLER, Further studies on Emden's and similar differential equations. *Quart. J. Math.*, **2** (1931), 259–288.
- [11] M. GHERGU; V. LISKEVICH; Z. SOBOLEV, Singular solutions for second-order non-divergence type elliptic inequalities in punctured balls. *J. Anal. Math.* **123** (2014), 251–279.
- [12] D. GILBARG; N. S. TRUDINGER, Elliptic partial differential equations of second order, second edition, 222, Springer-Verlag, 1993.
- [13] P. Han, Asymptotic behavior of solutions to semilinear elliptic equations with Hardy potential, *Proc. Amer. Math. Soc.*, **135** (2), (2006), 365–372.
- [14] Z. C. HAN, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent. *A. I. H. P* (1991) p.159–174.
- [15] E. HILLE, Sortie aspects of the Thomas-Fermi equation, *J. Anal. Math.*, **23** (1970), 147–170.
- [16] E. JANNELLI, The role played by space dimension in elliptic critical problems. *J. Differential Equations* **156** (1999), no. 2, 407–426.
- [17] M. K. KWONG; J. B. MCLEOD; L. A. PELETIER; W. C. TROY, On ground state solutions of $-\Delta u = u^p - u^q$, *J. Differential Equations*, **95**, (1992), 218–239.
- [18] F. MERLE; L. PELETIER, Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth. I. The radial case. *Arch. Rational Mech. Anal.* **112** (1990), no. 1, 1–19.
- [19] F. MERLE; L. PELETIER, Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth. II. The non-radial case. *J. Funct. Anal.* **105** (1992), no. 1, 1–41.
- [20] M. RAMASWAMY; S. SANTRA, Uniqueness and profile of positive solutions of a critical exponent problem with Hardy potential. *J. Differential Equations* **254** (2013), 4347–4372.
- [21] A. SOMMERFELD, Asymptotische integration der differential-gleichung des Thomas-Fermischen atoms, *Zeitschrift für Physik*, **78** (1932), 283–308.

- [22] S. TERRACINI, On the positive entire solutions to a class of equations with singular coefficients and critical exponents, *Adv. Differential Equations*, 1 (1996), no 2, 241–264.
- [23] J. L. VÁZQUEZ, An a priori interior estimate for the solutions of a nonlinear problem representing weak diffusion. *Nonlinear Anal.* 5 (1981), no. 1, 95–103.
- [24] J. L. VÁZQUEZ; L. VÉRON, Isolated singularities of some semilinear elliptic equations. *J. Differential Equations* 60 (1985), no. 3, 301–321.
- [25] VÉRON, LAURENT, Comportement asymptotique des solutions d'équations elliptiques semi-linéaires dans \mathbb{R}^N . *Ann. Mat. Pura Appl.* (4) 127 (1981), 25–50.
- [26] C. L. XIANG, Gradient estimates for solutions To quasilinear elliptic equations with critical Sobolev growth and Hardy potential. *arXiv:1502.03968*.

M. BHAKTA, DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, DR. HOMI BHABHA ROAD, PUNE 411008, INDIA
E-mail address: `mousomi@iiserpune.ac.in`

S. SANTRA, DEPARTMENT OF BASIC MATHEMATICS, CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS, GUANAJUATO, MÉXICO
E-mail address: `sanjiban@cimat.mx`